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Hyperbolic Quasilinear Navier–Stokes Equations in $\mathbb{R}^2$

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Abstract
We consider a hyperbolic quasilinear version of the Navier–Stokes equations in $\mathbb{R}^2$, arising from using a Cattaneo type law instead of a Fourier law. These equations, which depend on a parameter ε, are a way to avoid the infinite speed of propagation which occurs in the classical Navier–Stokes equations. We first prove the existence and uniqueness of solutions to these equations, and then exhibit smallness assumptions on the data, under which the solutions are global in time. In particular, these smallness assumptions disappear when ε vanishes, accordingly to the fact that the solutions of the 2D Navier–Stokes equations are global.

1 Introduction

The Navier–Stokes equations, which govern the Newtonian fluids, have been a tremendous topic of research since their introduction in the 30s. However, one physical issue of these equations comes from the fact that the information is propagated with infinite speed. In order to avoid this non physical feature, a solution consists in considering a hyperbolic perturbation of the equations, depending on a small parameter $\varepsilon > 0$, for which the information propagates with finite speed. In the literature, there exist several examples of such hyperbolic versions of the Navier–Stokes equations, obtained through different approaches. For instance, in [3], Brenier, Natalini and Puel have introduced a hyperbolic system of equations, based on a relaxation approximation of the incompressible Navier–Stokes equations, following the scheme described by Jin and Xin in [11]. Their existence and uniqueness result has been then improved by Paicu and Raugel in [13,14] and Hachicha in [10]. The way these authors introduce their system of equations is a smart way to take advantage of methods which are usually devoted to the study of numerical schemes, and which can be applied to every conservation laws. In this article, we take the problem from another point of view. Indeed, instead of directly approximating the Navier–Stokes equations, we prefer considering an

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alternative physical model, and then derive an approximate system of equations from it. This way, the momentum equations we obtain are not only a mathematical approximation of the Navier–Stokes equations, but also have a physical meaning, even if ε does not vanish. More precisely, in this paper, we consider the system of equations which is obtained by replacing the classical Fourier law, leading to the Navier–Stokes equations, by a Cattaneo law. Initially, this law was proposed by Cattaneo and other authors in the late 40s, at first as a hyperbolic approximation to the heat equation (see for instance [6,7,21]). Later, this idea was extended to fluid-dynamics, notably by Carrassi and Morro [5] and Carbonaro and Rosso [4]. More precisely, consider $u = u(t, x) \in \mathbb{R}^d$, $d = 2, 3$ to be the velocity field of an incompressible fluid with constant density 1, $t > 0$ and $x \in \mathbb{R}^d$ being the time and space variables, respectively. In this case, since the fluid is assumed to be incompressible, $\operatorname{div} u = 0$ and the momentum equations write

$$\partial_t u + u.\nabla u = f + \operatorname{div}(\sigma), \tag{1.1}$$

where f is the external forcing term and σ denotes the stress tensor. The Fourier law, which governs the Newtonian fluids, is then given by the stress tensor definition

$$\sigma(t) = -p(t)I + \nu\left(\nabla u(t) + (\nabla u)^t(t)\right), \tag{1.2}$$

where p is the pressure of the fluid and $\nu > 0$ is the kinetic viscosity. Replacing σ by the identity (1.2) in (1.1) gives the classical incompressible Navier–Stokes equations

$$\begin{aligned} &\partial_t u - \nu\Delta u + u.\nabla u = -\nabla p + f, \\ &\operatorname{div} u = 0. \end{aligned} \tag{1.3}$$

In this article, we are interested in the equations obtained when we use a Cattaneo type law instead of the Fourier law in the equations of conservation of momentum (1.1). More precisely, the stress tensor we consider is defined as the solution of the differential equation

$$\sigma(t) + \varepsilon\partial_t\sigma(t) = -p(t)I + \nu\left(\nabla u(t) + (\nabla u)^t(t)\right). \tag{1.4}$$

Notice that the left hand side of (1.4) is obtained by performing the first order Taylor approximation of $\sigma(t + \varepsilon)$, where $\varepsilon > 0$ is a delay parameter. If ε vanishes, the Fourier law is recovered and we go back to (1.3). By using the Cattaneo law (1.4) in the equation of conservation of momentum and applying $(1 + \varepsilon\partial_t)$ to the Eq. (1.1), we obtain the following hyperbolic momentum equation

$$\varepsilon\partial_t^2 u + \partial_t u - \nu\Delta u + u.\nabla u + \varepsilon\partial_t u.\nabla u + \varepsilon u.\nabla\partial_t u = -\nabla p + f + \varepsilon\partial_t f. \tag{1.5}$$

This equation is quite similar to what was studied in [3,10,13,14] from relaxation schemes, but includes the two additional non-linear terms $\partial_t u.\nabla u$ and $u.\nabla\partial_t u$, which may be tricky to estimate. In particular, the minimum regularity needed to establish the existence of solutions of (1.5) is higher than what is shown for the relaxation-based equations (for instance in [13]). Finally, we assume that $\nu = 1$ and consider the following Cauchy problem on $u_\varepsilon = u_\varepsilon(\tau, x)$:

$$\begin{aligned} \varepsilon\partial_\tau^2 u_\varepsilon + \partial_\tau u_\varepsilon - \Delta u_\varepsilon + u_\varepsilon.\nabla u_\varepsilon + \varepsilon\partial_\tau u_\varepsilon.\nabla u_\varepsilon + \varepsilon u_\varepsilon.\nabla\partial_\tau u_\varepsilon &= -\nabla p_\varepsilon + f_\varepsilon, \\ \operatorname{div} u_\varepsilon &= 0, \\ (u_\varepsilon, \partial_\tau u_\varepsilon)(0, y) &= (u_{0,\varepsilon}, u_{1,\varepsilon})(y), \end{aligned} \tag{1.6}$$

where f_ε stands for $f + \varepsilon\partial_\tau f$.

The purpose of this paper is to show the existence and uniqueness of solutions to (1.6) in the whole space $\mathbb{R}^2$ for sufficiently regular initial data $(u_{0,\varepsilon}, u_{1,\varepsilon})$. Previously, Racke and Saal have addressed the same problem in [15,16] and have shown the existence and uniqueness of local solutions to (1.6) in $\mathbb{R}^d$, $d = 2, 3$, when the initial data belong to the Sobolev spaces $H^{m+2}(\mathbb{R}^d)^d \times H^{m+1}(\mathbb{R}^d)^d$, $m > \frac{d}{2}$. The method they used is based on a priori estimates on the solutions of a linearised and regularised version of (1.6). Then, a fix-point method allows to conclude to the existence and uniqueness of solutions to the initial system (1.6). In addition, if the initial data are sufficiently small in a very regular Sobolev space, the solutions are global in time (see [16, Theorem 6.1]). In the three dimensional case, Schöwe in [17–19] and Abdelhedi in [1] have improved the global existence result. Notably, Abdelhedi has given a condition which allows to consider global solutions when the initial data are small enough in the space $H^4(\mathbb{R}^3)^3 \times H^3(\mathbb{R}^3)^3$. In addition, she has shown that if ε is close to 0, then the solution of (1.6) is close to the solution to the classical Navier–Stokes equations. In what follows, we will show that, in dimension 2, there exist solutions to the system (1.6), when the initial data belong to $H^{2+\eta}(\mathbb{R}^2)^2 \times H^{1+\eta}(\mathbb{R}^2)^2$, for all $1 > \eta > 0$. Furthermore, if the initial data satisfy a ε-dependent smallness assumption, then these solutions are global in time. Another improvement to the works of Racke, Saal and Schöwe comes from the fact that the smallness assumption on the data disappears when ε goes to 0, in accordance with the global existence of solutions of the Navier–Stokes equations in the two dimensional case [9,12]. In order to obtain the local existence of solutions, the method we use is based on a Friedrichs scheme. More precisely, we consider a sequence of systems, whose solutions belong to more regular spaces. Then, by passing to the limit, we show that the sequence of solutions converges to a solution of (1.6). The global existence for small data is obtained through energy estimates, combined with a frequency decomposition of the solution. Notice that this paper focuses on the two dimensional case only. Actually, the energy estimates we make on the nonlinear terms are very specific to the dimension 2, and use sharp Sobolev injections which do not extend to the 3D case. However, by adapting the method accordingly, we may also obtain the same kind of theorems for the 3D setting, and probably improve the results of [1]. Anyway, this would deserve a specific study, which we do not address here.

The outline of this paper is as follows. In Sect. 2, we define the mathematical formalism of the problem, and state the main local and global existence theorems. The Sect. 3 is devoted to the proof of the local existence theorem, for which we use a Friedrichs scheme, that is to say of sequence of regularised systems, whose limit is (1.6). We also prove in this section the uniqueness of the solutions, as well as their time-continuity, which is achieved via the decomposition of the solution u as the sum $u = u_1 + u_2$, where u_1 is continuous in time and more regular than u and u_2 remains small. Finally, in Sect. 4, we show that if the initial data satisfy a ε-dependent inequality, then the solution obtained in Sect. 3 is actually global in time. This is done by writing the solution as the sum of a low-frequency term and a high-frequency term. Then, separate energy estimates on those two terms allow to conclude to the infinite time of existence, if the initial data are small enough.

2 Notations and Main Statements

In this section, after introducing a scaled ε-independent version of (1.6) and giving some recalls about the Sobolev spaces, we define precisely what a solution of (1.6) is and state the main existence and uniqueness theorems of this article. First of all, in order to eliminate the parameter ε-dependency, we perform the rescaling

$$\begin{aligned} u_\varepsilon(\tau, y) &= \frac{1}{\sqrt{\varepsilon}} u\left(\frac{\tau}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}\right) \ , \ f_\varepsilon(\tau, y) = \frac{1}{\varepsilon\sqrt{\varepsilon}} f\left(\frac{\tau}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}\right) \\ p_\varepsilon(\tau, y) &= \frac{1}{\varepsilon} p\left(\frac{\tau}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}\right), \end{aligned} \tag{2.1}$$

and we set

$$t = \frac{\tau}{\varepsilon} \ , \quad x = \frac{y}{\sqrt{\varepsilon}} \ .$$

This scaling transforms the ε-dependent Eq. (1.6) into the following system of equations with initial data which depend on ε:

$$\begin{aligned} &\partial_t^2 u + \partial_t u - \Delta u + u.\nabla u + \partial_t u.\nabla u + u.\nabla \partial_t u = -\nabla p + f, \\ &\text{div } u = 0, \\ &(u, \partial_t u)(0, x) = (\sqrt{\varepsilon} u_{0,\varepsilon}(\sqrt{\varepsilon} x), \varepsilon^{3/2} u_{1,\varepsilon}(\sqrt{\varepsilon} x)) \equiv (u_0, u_1) \ . \end{aligned} \tag{2.2}$$

Throughout this paper, we will establish the existence and uniqueness of solutions to the system (2.2), and then deduce similar results on the system (1.6), by performing the scaling (2.1) backwards. The main advantage of this technique appears when dealing with the global existence result. Indeed, the smallness assumption on the initial data will be computed for the solutions of (2.2), resulting into a ε-independent condition. By going back to (1.6) through the inverse scaling associated to (2.1), we will then be able to exhibit the ε-dependency of the smallness assumption on the initial data of (1.6).

In order to state our main theorem, we now need to introduce several functions spaces. Let first recall the definition of the non-homogeneous Hilbert spaces $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, given by

$$H^s(\mathbb{R}^2) = \{u \in \mathcal{S}'(\mathbb{R}^2) \,|\, \hat{u} \in L^2_{loc}(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \ < +\infty\} \ .$$

equipped with the norm

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \ ,$$

where $\hat{u}$ is the Fourier transform of u, given by

$$\hat{u}(\xi) = \int_{\mathbb{R}^2} u(x) e^{-ix.\xi} dx \ .$$

In what follows, $(\cdot, \cdot)_s$ denotes the scalar product associated to $H^s(\mathbb{R}^2)$. We will also use the usual $L^2(\mathbb{R}^2)$ scalar product as well as the L^2-norm, which will be denoted $(\cdot, \cdot)$ and $\|\cdot\|$ respectively. Together with the classical Sobolev spaces, we will need some properties of the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, whose definition is given by

$$\dot{H}^s(\mathbb{R}^2) = \{u \in \mathcal{S}'(\mathbb{R}^2) \,|\, \hat{u} \in L^2_{loc}(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \ < +\infty\} \ ,$$

The $\dot{H}^s(\mathbb{R}^2)$ corresponding seminorm writes

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \ ,$$

and $(.,.)_{\dot{H}^s}$ denotes the associated scalar product. Since the solutions we consider are divergence free vector fields, we introduce the following distribution space

$$\mathcal{V} = \{u \in \mathcal{D}(\mathbb{R}^2)^2 : \operatorname{div} u = 0\},$$

as well as its L^2 closure $H = \overline{\mathcal{V}}^{L^2}$. Finally, the solutions that we exhibit in this article exist in the following Sobolev spaces of divergence-free vector fields, defined by

$$H^s_\sigma = H \cap H^s(\mathbb{R}^2)^2.$$

Let us now give the precise definition of what a solution to the system (2.2) is. In this paper, we show the existence of solutions in a weak sense, that is to say the first equality of (2.2) is satisfied through a dual formulation. It is stated in the definition below.

Definition 2.1 For $\eta > 0$, we call a solution of (2.2) with initial data $(u_0, u_1) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ a couple $(u, \partial_t u) \in C^0_w\left([0,T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right)$ such that $(u, \partial_t u)_{|t=0} = (u_0, u_1)$ and, for all $t \in [0,T]$ and $\varphi \in L^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \left(\partial_t^2 u + \partial_t u - \Delta u\right)(t,x)\varphi(x)dx \\ + \int_{\mathbb{R}^2} \mathbb{P}\left(u.\nabla u + \partial_t u.\nabla u + u.\nabla\partial_t u\right)(t,x)\varphi(x)dx = \int_{\mathbb{R}^2} \mathbb{P} f(t,x)\varphi(x)dx, \tag{2.3}$$

where $\mathbb{P} : L^2(\mathbb{R}^2)^2 \to H$ is the classical Leray projector on $\mathbb{R}^2$.

With these definitions at hand, we are able to state the main results of this paper. We will first prove the following theorem of local existence and uniqueness of solutions.

Theorem 2.1 *Let $0 < \eta < 1$ be given, $(u_0, u_1) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ and $f \in C^0\left([0,+\infty), H^{1+\eta}\right)$. There exist a positive time T and a unique local solution $u \in C^0\left([0,T], H^{2+\eta}_\sigma\right) \cap C^1\left([0,T], H^{1+\eta}_\sigma\right) \cap C^2\left([0,T], H^{\eta}_\sigma\right)$ to the system (2.2), in the sense of Definition 2.1.*

According to this theorem, we cannot expect the existence of solutions if the initial data have a lower regularity than $H^2 \times H^1$. As said earlier, it is one of the differences with the relaxation-based systems, for which the existence of solutions is established with initial data in $H^1 \times L^2$ only (see [10,13],). The second main result of this paper establishes the global existence of solutions of (2.2) if the initial data are small enough. Then, going back to the ε-dependent system (1.6), we will see that the smallness assumption on the initial data can be reduced to the choice of a sufficiently small ε.

Theorem 2.2 *Let $0 < \eta < 1$ be given. There exist two positive constants K_0 and K_1 such that, for any initial data $(u_0, u_1) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ and for any forcing term $f \in C^0([0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^2((0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^1((0,+\infty), L^2(\mathbb{R}^2)^2)$ satisfying the condition*

$$B_0 + B_0^{1/2}\left(\|u_0\|^2_{L^2} + \|u_1\|^2_{L^2} + \|f\|^2_{L^1((0,+\infty),L^2)}\right)^{1/2} \leq K_0, \tag{2.4}$$

where

$$B_0 = \|\nabla u_0\|^2_{L^2} + \|u_0\|^2_{\dot{H}^{2+\eta}} + \|\nabla u_1\|^2_{L^2} + \|u_1\|^2_{\dot{H}^{1+\eta}} + \|f\|^2_{L^2((0,+\infty),H^{1+\eta})},$$

there exists a unique global solution $u \in C^0\left([0,+\infty), H^{2+\eta}_\sigma\right) \cap C^1\left([0,+\infty), H^{1+\eta}_\sigma\right) \cap C^2\left([0,+\infty), H^{\eta}_\sigma\right)$ *to the system* (2.2) *which satisfies, for any* $t \geq 0$,

$$\begin{aligned}\|u(t)\|_{2+\eta} + \|\partial_t u(t)\|_{1+\eta} \leq & K_1\big(\|u_0\|_{2+\eta} + \|u_1\|_{1+\eta} \\ & + \|f\|_{L^2((0,+\infty),H^{1+\eta})} + \|f\|_{L^1((0,+\infty),L^2)}\big).\end{aligned} \tag{2.5}$$

The two Theorems 2.1 and 2.2 can be translated in the initial variables, giving the existence and uniqueness of solutions to the system (1.6). First, by using the scaling (2.1) backwards, the Theorem 2.1 directly gives the following one.

Theorem 2.3 *Let* $0 < \eta < 1$ *be given,* $(u_{0,\varepsilon}, u_{1,\varepsilon}) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ *and* $f_\varepsilon \in C^0\left([0,+\infty), H^{1+\eta}\right)$. *There exist a positive time* $T = T(\varepsilon)$ *and a unique local solution* $u_\varepsilon \in C^0\left([0,T], H^{2+\eta}_\sigma\right) \cap C^1\left([0,T], H^{1+\eta}_\sigma\right) \cap C^2\left([0,T], H^{\eta}_\sigma\right)$ *to the system* (1.6).

We then derive a global existence theorem for the system (1.6), from the Theorem 2.2. To do so, we need to translate the smallness assumption (2.4) in terms of the initial variables. A straightforward computation gives the following equalities:

$$\begin{aligned}\|u\|_{\dot{H}^s_x} =& \varepsilon^{\frac{s}{2}}\|u_\varepsilon\|_{\dot{H}^s_y}\ , \quad \forall s \in \mathbb{R}\ ,\\ \|u_t\|_{\dot{H}^s_x} =& \varepsilon^{1+\frac{s}{2}}\|u_{\varepsilon,\tau}\|_{\dot{H}^s_y}\ , \quad \forall s \in \mathbb{R}\ ,\\ \|u\|_{L^p_x} =& \varepsilon^{\frac{1}{2}-\frac{1}{p}}\|u_\varepsilon\|_{L^p_y}\ , \quad \forall p \geq 1\ ,\end{aligned} \tag{2.6}$$

and in particular

$$\|u\|_{L^2_x} = \|u_\varepsilon\|_{L^2_y}\ ,\ \|\nabla u\|_{L^2_x} = \varepsilon^{\frac{1}{2}}\|\nabla u_\varepsilon\|_{L^2_y}\ ,\ \|u_t\|_{L^2_x} = \varepsilon\|u_{\varepsilon,\tau}\|_{L^2_y}\ . \tag{2.7}$$

Likewise, we also obtain the following equality, for $p \geq 1$,

$$\|f\|_{L^p(\dot{H}^s_x)} = \varepsilon^{\frac{s}{2}+1-\frac{1}{p}}\|f_\varepsilon\|_{L^p(\dot{H}^s_y)}\ , \quad \forall p \geq 1\ , \quad \forall s \in \mathbb{R}\ . \tag{2.8}$$

Using (2.6), (2.7) and (2.8), we at once deduce the following global existence result for the equations (1.6).

Theorem 2.4 *Let* $0 < \eta < 1$ *and* $\varepsilon > 0$ *be given. There exist two positive constants* K_0 *and* K_1 *such that, for any initial data* $(u_{0,\varepsilon}, u_{1,\varepsilon}) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ *and any forcing term* $f_\varepsilon \in C^0([0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^2((0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^1((0,+\infty), L^2(\mathbb{R}^2)^2)$ *satisfying the condition*

$$B_{0,\varepsilon} + B_{0,\varepsilon}^{1/2}\big(\|u_{0,\varepsilon}\|^2_{L^2} + \varepsilon^2\|u_{1,\varepsilon}\|^2_{L^2} + \|f_\varepsilon\|^2_{L^1((0,+\infty),L^2)}\big)^{1/2} \leq K_0\ , \tag{2.9}$$

where

$$\begin{aligned}B_{0,\varepsilon} =& \varepsilon\big(\|\nabla u_{0,\varepsilon}\|^2_{L^2} + \varepsilon^{1+\eta}\|u_{0,\varepsilon}\|^2_{\dot{H}^{2+\eta}} + \varepsilon^2\|\nabla u_{1,\varepsilon}\|^2_{L^2} + \varepsilon^{2+\eta}\|u_1\|^2_{\dot{H}^{1+\eta}}\\ & + \|f_\varepsilon\|^2_{L^2((0,+\infty),H^{1+\eta})}\big)\ ,\end{aligned}$$

there exists a unique global solution $u_\varepsilon \in C^0\left([0,+\infty), H^{2+\eta}_\sigma\right) \cap C^1\left([0,+\infty), H^{1+\eta}_\sigma\right) \cap C^2\left([0,+\infty), H^{\eta}_\sigma\right)$ *to the system* (1.6) *which satisfies, for any* $t \geq 0$,

$$\begin{aligned} &\|u_\varepsilon(\tau)\|_{L^2} + \varepsilon\|\partial_\tau u_\varepsilon(\tau)\|_{L^2} + \varepsilon^{\frac{2+\eta}{2}}\left(\|u_\varepsilon(\tau)\|_{\dot H^{2+\eta}} + \varepsilon^{\frac12}\|\partial_\tau u_\varepsilon(\tau)\|_{\dot H^{1+\eta}}\right)\\ &\leq K_1\Big[\|u_{0,\varepsilon}\|_{L^2} + \varepsilon\|u_{1,\varepsilon}\|_{L^2} + \varepsilon^{\frac{2+\eta}{2}}(\|u_{0,\varepsilon}\|_{\dot H^{2+\eta}} + \varepsilon^{\frac12}\|u_{1,\varepsilon}\|_{\dot H^{1+\eta}}) \\ &\quad + \|f_\varepsilon\|_{L^1((0,+\infty),L^2)} + \varepsilon^{\frac12}\|f_\varepsilon\|_{L^2((0,+\infty),L^2)} + \varepsilon^{\frac{2+\eta}{2}}\|f_\varepsilon\|_{L^2((0,+\infty),\dot H^{1+\eta})}\Big]. \end{aligned} \tag{2.10}$$

In particular, the smallness assumption 2.9 disappears when ε goes to 0. Consequently, as a direct consequence of Theorem 2.4, we can conclude that, for every initial data and forcing term satisfying the conditions of the local existence Theorem 2.3, if ε is small enough, then the solution is actually global in time. This is stated in the next Corollary.

Corollary 2.1 *Let* $0 < \eta < 1$ *be given. For any initial data* $(u_{0,\varepsilon}, u_{1,\varepsilon}) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ *and any forcing term* $f_\varepsilon \in C^0([0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^2((0,+\infty), H^{1+\eta}(\mathbb{R}^2)^2) \cap L^1((0,+\infty), L^2(\mathbb{R}^2)^2)$*, there exists a constant* $\varepsilon_0 > 0$ *such that, if* $\varepsilon < \varepsilon_0$*, then there exists a unique global solution to the system* (1.6)*, satisfying the properties of Theorem* 2.4.

Notice that, in [16], is conjectured that solutions to the system (1.6) with large initial data may blow up at finite time. The previous corollary does not contradict this conjecture, but shows that if a blow up condition exists, it is ε-dependent. In other words, the definition of a large initial data is obviously relative to ε, as well as it is the case for the definition of a small initial data, given through the inequality (2.9). The remaining of this article is devoted to the Proofs of the Theorems 2.1 and 2.2 .

3 Proof of Theorem 2.1: Existence of Local Solutions

In this Section, we show Theorem 2.1. The method we use to prove the local existence of solutions of (2.2) is based on a Friedrichs scheme, consisting in defining a regularised system which depends on a parameter $n \in \mathbb{N}$. Since the solutions u_n of these equations are very regular, we are then able to perform energy estimates involving the Sobolev norms of u_n. Then, by letting n go to infinity, we prove that u_n converges to a solution of (2.2), which also satisfies the same energy estimates. Finally, we establish the time continuity of the obtained solution.

3.1 Basic Auxiliary Properties

In order to introduce a regular version of (2.2), we will use the following mollifier operator Π_n, for $n \in \mathbb{N}$, which is the spectral cut-off function defined by

$$\Pi_n(u) = \mathcal{F}^{-1}\left(\chi_{[0,n]}\hat u\right), \tag{3.1}$$

where $\mathcal{F}$ denotes the Fourier transform, $\hat u = \mathcal{F}u$ and $\chi_{[0,n]}$ is the cut-off function, given by

$$\chi_{[0,n]}(\xi) = 1\,, \quad \forall\,|\xi| \leq n\,, \quad \text{and} \quad \chi \equiv 0\,, \quad \forall\,|\xi| > n\,.$$

Due to the well-known properties of the Fourier transform, the following properties hold:

1. $\Pi_n^2 = \Pi_n$.

2. Π_n commutes with the derivatives.
3. Π_n is a self-adjoint operator with respect to the L^2-scalar product.

The most important properties of Π_n are the smoothing properties, which we recall in the next lemma.

Lemma 3.1 *The operator Π_n satisfies the following properties:*

(1) There exists a constant $C > 0$ such that we have, for any $u \in H^s$ and $s \le \sigma$,

$$\begin{aligned} \|\Pi_n u\|_\sigma &\le C n^{\sigma - s} \|\Pi_n u\|_s \le C n^{\sigma - s}\|u\|_s \\ \|\Pi_n u\|_{\dot{H}^\sigma} &\le C n^{\sigma - s} \|\Pi_n u\|_{\dot{H}^s} \le C n^{\sigma - s}\|u\|_{\dot{H}^s} \ . \end{aligned} \tag{3.2}$$

Thus, for any $s \in \mathbb{R}$, Π_n is a continuous operator from $H^s(\mathbb{R}^2)$ into $\bigcap_{k\in\mathbb{Z}} H^k(\mathbb{R}^2)$.

(2) Likewise, there exists a constant $c > 0$ such that, for any $u \in H^s$ and $\sigma \le s$,

$$c n^{s-\sigma} \|(I - \Pi_n)u\|_{\dot{H}^\sigma} \le \|(I - \Pi_n)u\|_{\dot{H}^s} \ . \tag{3.3}$$

(3) If u belongs to $H^s(\mathbb{R}^2)$, then $\Pi_n(u) \to u$ strongly in $H^s(\mathbb{R}^2)$, when n goes to infinity.

Proof The above inequalities are straightforward consequences of the definition of Π_n. Indeed, for example, we have, for $s \le \sigma$,

$$\begin{aligned} \|\Pi_n u\|^2_{\dot{H}^\sigma} &= \int_{\mathbb{R}^2} |\xi|^{2\sigma} \chi_{[0,n]}(\xi)|\hat{u}|^2 d\xi = \int_{|\xi|\le n} |\xi|^{2\sigma} |\hat{u}|^2 d\xi \\ &\le n^{2(\sigma - s)} \int_{\mathbb{R}^2} |\xi|^{2s} \chi_{[0,n]}(\xi)|\hat{u}|^2 d\xi \ , \end{aligned} \tag{3.4}$$

which implies the second inequality in (3.2). The other inequalities are proved in a similar way. □

In order to perform energy estimates on the solutions of the regularised system, we also need to bound the product of two vectors. These elementary products laws are given in the next proposition.

Proposition 3.1 *(1) For any $s \ge 0$, there exists a positive constant $C_1 = C_1(s)$, such that, for any $u_1, u_2 \in L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$,*

$$\|u_1 u_2\|_s \le C_1(s)\big(\|u_1\|_{L^\infty}\|u_2\|_s + \|u_2\|_{L^\infty}\|u_1\|_s\big). \tag{3.5}$$

In particular, if $s > 1$, $H^s(\mathbb{R}^2)$ is an algebra and there exists a positive constant $C_2 = C_2(s)$ such that, for any $u_1, u_2 \in H^s(\mathbb{R}^2)$,

$$\|u_1 u_2\|_s \le C_2(s)\|u_1\|_s \|u_2\|_s. \tag{3.6}$$

(2) Likewise, for any $s \ge 0$ and u_1, u_2 in $L^\infty(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$,

$$\|u_1 u_2\|_{\dot{H}^s} \le C_1(s)\big(\|u_1\|_{L^\infty}\|u_2\|_{\dot{H}^s} + \|u_2\|_{L^\infty}\|u_1\|_{\dot{H}^s}\big). \tag{3.7}$$

(3) For any $s \ge 0$, there exists a positive constant $C_3 = C_3(s)$ such that, for any $u_1, u_2 \in (L^\infty(\mathbb{R}^2))^2 \cap (H^s(\mathbb{R}^2))^2$ such that $\operatorname{div} u_1 = 0$, the following inequality holds:

$$\|u_1 \cdot \nabla u_2\|_{s-1} \le C_3(s)\big(\|u_1\|_{L^\infty}\|u_2\|_s + \|u_2\|_{L^\infty}\|u_1\|_s\big). \tag{3.8}$$

(4) *Likewise, for any $s \geq 1$, there exists a positive constant $C_4 = C_4(s)$ such that, for any $u_1, u_2 \in (L^\infty(\mathbb{R}^2))^2 \cap (\dot{H}^s(\mathbb{R}^2))^2$ such that $\operatorname{div} u_1 = 0$, the following inequality holds:*

$$\|u_1 \cdot \nabla u_2\|_{\dot{H}^{s-1}} \leq C_4(s)\big(\|u_1\|_{L^\infty}\|u_2\|_{\dot{H}^s} + \|u_2\|_{L^\infty}\|u_1\|_{\dot{H}^s}\big). \tag{3.9}$$

Proof The proof of the inequalities (3.5) and (3.7) can be found in [8] and [2]. The inequality (3.8) is a direct consequence of (3.5) and of the remark that $u_1 \cdot \nabla u_2 = \nabla(u_1 \otimes u_2)$, if both terms make sense and if u_1 is divergence-free. The inequality (3.9) is proved in [13, Annexe A]. □

Finally, for $s \geq 0$, we introduce the operator $J_s = (I - \Delta)^{\frac{s}{2}}$, which is defined by

$$J_s u = \mathcal{F}^{-1}\left(\left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)\right) .$$

In particular, we notice that, for all $s \in \mathbb{R}$,

$$\|u\|_s = \|J_s u\|_{L^2} \ .$$

The following commutator property of J_s will be needed when performing a priori estimates on the solutions of (2.2).

Lemma 3.2 *Let $s > 1$. There exists a constant $C > 0$ such that, for any $u \in H^{s+1}(\mathbb{R}^2)$ and $v \in H^{s-1}(\mathbb{R}^2)$, we have*

$$\|[J_s, u]\, v\|_{L^2} \leq C\, \|J_s \nabla u\|_{L^2}\, \|J_{s-1} v\|_{L^2} . \tag{3.10}$$

Proof The inequality (3.10) is obtained by adapting the proof of [20, Lemma 2.4] to the case of the dimension 2. □

3.2 Regularised System and a Priori Estimates

We recall that $\mathbb{P} : L^2(\mathbb{R}^2)^2 \to H$ denotes the classical Leray orthogonal projector from $L^2(\mathbb{R}^2)^2$ onto H, given by the non local pseudo-differential operator

$$\mathbb{P} = I - \nabla(\Delta)^{-1}\mathrm{div} .$$

We now introduce a regularised version of the system (2.2), which we obtain through the cut-off operator Π_n and the Leray projector. It is given by

$$\begin{aligned} &\partial_t^2 u_n + \partial_t u_n - \Delta \Pi_n u_n + \mathbb{P}\Pi_n\left(\Pi_n u_n.\nabla \Pi_n u_n\right) + \mathbb{P}\Pi_n\left(\Pi_n \partial_t u_n.\nabla \Pi_n u_n\right) \\ &\quad + \mathbb{P}\Pi_n\left(\Pi_n u_n.\nabla \Pi_n \partial_t u_n\right) = \mathbb{P}\Pi_n(f), \\ &(u_n(0), \partial_t u_n(0)) = (\Pi_n(u_0), \Pi_n(u_1)). \end{aligned} \tag{3.11}$$

By defining the vector $\vec{V}_n = (V_1, V_2) \equiv (u_n, \partial_t u_n)$, we can rewrite the above second order system (3.11) as the following first order system

$$\begin{aligned} \partial_t \vec{V}_n &= \vec{F}(t, V_n), \\ \vec{V}_n(0) &= (\Pi_n(u_0), \Pi_n(u_1)) , \end{aligned} \tag{3.12}$$

where $\vec{F} = (F_1, F_2)$, and

$$\begin{aligned} F_1(t, \vec{V}) &= V_2, \\ F_2(t, \vec{V}) &= -V_2 + \Delta \Pi_n V_1 - \mathbb{P}\Pi_n\left(\Pi_n V_1.\nabla \Pi_n V_1\right) - \mathbb{P}\Pi_n\left(\Pi_n V_2.\nabla \Pi_n V_1\right) \\ &\quad - \mathbb{P}\Pi_n\left(\Pi_n V_1.\nabla \Pi_n V_2\right) + \mathbb{P}\Pi_n(f(t)). \end{aligned}$$

For all $\eta > 0$, we notice that $\vec{F}$ is a continuous map from $[0, +\infty) \times \Pi_n(H_\sigma^{2+\eta}) \times \Pi_n(H_\sigma^{1+\eta})$ to $\Pi_n(H_\sigma^{2+\eta}) \times \Pi_n(H_\sigma^{1+\eta})$. In addition, $\vec{F}$ is a locally-Lipschitzian function in $V \in \Pi_n(H_\sigma^{2+\eta}) \times \Pi_n(H_\sigma^{1+\eta})$, for every $0 < \eta < 1$. Consequently, the Cauchy-Lipschitz Theorem implies that, for every $(u_0, u_1) \in H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}$, there exists a positive maximal time T_n and a unique solution $(u_n, \partial_t u_n) \in C^1\left([0, T_n], \Pi_n(H_\sigma^{2+\eta}) \times \Pi_n(H_\sigma^{1+\eta})\right)$ to the system (3.11). Furthermore, if $T_n < +\infty$, then $\|(u_n, \partial_t u_n)\|_{H^{2+\eta}\times H^{1+\eta}} \longrightarrow +\infty$ when $t \to T_n$. We notice that $(\Pi_n(u_n), \Pi_n(\partial_t u_n))$ is also a solution to (3.11), and, since this solution is unique, it comes

$$\Pi_n(u_n) = u_n, \quad \text{and} \quad \Pi_n(\partial_t u_n) = \partial_t u_n.$$

Thus, $(u_n, \partial_t u_n)$ satisfies the equation

$$\partial_t^2 u_n + \partial_t u_n - \Delta u_n + \mathbb{P}\Pi_n\left(u_n.\nabla u_n\right) + \mathbb{P}\Pi_n\left(\partial_t u_n.\nabla u_n\right) + \mathbb{P}\Pi_n\left(u_n.\nabla \partial_t u_n\right) = \mathbb{P}\Pi_n(f). \tag{3.13}$$

The aim of this section is to show that, for all $n \in \mathbb{N}$, there exists a positive time $T \in (0, T_n]$, independent of n such that $(u_n, \partial_t u_n)$ is bounded in the $H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}$-norm on $[0, T]$, uniformly with respect to n. To this end, we introduce the following energy functional

$$E_n(t) = \frac{1}{2}\left(\|u_n + \partial_t u_n\|_{1+\eta}^2 + \|\partial_t u_n\|_{1+\eta}^2\right) + \|\nabla u_n\|_{1+\eta}^2.$$

We notice that $E_n(t)$ is equivalent to $\|u_n(t)\|_{2+\eta} + \|\partial_t u_n(t)\|_{1+\eta}$. Indeed, for every $(v_0, v_1) \in (H^{2+\eta} \times H^{1+\eta})$, the following inequalities hold:

$$\begin{aligned} \frac{1}{8}\left(\|v_0\|_{2+\eta}^2 + \|v_1\|_{1+\eta}^2\right) &\leq \frac{1}{2}\left(\|v_0 + v_1\|_{1+\eta}^2 + \|v_1\|_{1+\eta}^2\right) + \|\nabla v_0\|_{1+\eta}^2 \\ &\leq \frac{3}{2}\left(\|v_0\|_{2+\eta}^2 + \|v_1\|_{1+\eta}^2\right). \end{aligned} \tag{3.14}$$

Through a priori estimates on the solutions of (3.13), we will show that

$$\partial_t E_n(t) \leq G(t) E_n^2(t), \quad \text{for all} \quad t \in [0, T_n], \tag{3.15}$$

where G is a locally integrable function which depends on f. By integrating the previous inequality in time and taking into account the inequalities (3.14), we will then be able to obtain uniform bounds on the $(H^{2+\eta} \times H^{1+\eta})$-norm of $(u_n, \partial_t u_n)$, which are valid on a time interval $[0, T)$, where $T \in (0, T_n]$ is independent of n. In particular, this implies that the solution $(u_n, \partial_t u_n)$ exists as long as $0 \leq t \leq T$, for every $n \in \mathbb{N}$.

To obtain estimates of $(u_n(t), \partial_t u_n(t))$ in the $(H^{2+\eta} \times H^{1+\eta})$-norm, we begin by proving the following lemma. In what follows, the constant C denotes a positive constant which may change from one line to another.

Lemma 3.3 *There exists a positive constant $C > 0$, such that, for all $n \in \mathbb{N}$ and all $t \in [0, T_n)$,*

$$\partial_t E_n(t) + \frac{1}{4}\|\nabla u_n(t)\|_{1+\eta}^2 + \|\partial_t u_n(t)\|_{1+\eta}^2 \leq 2\|f(t)\|_{1+\eta}^2 + 1 + C E_n^2(t). \tag{3.16}$$

Proof We first apply the operator $J_{1+\eta}$ to the equation (3.13) and take the L^2-inner product of the resulting equation with $J_{1+\eta}(u + 2\partial_t u)$. We notice that the resulting equality makes sense since $(u_n, \partial_t u_n)$ is as regular as wanted. Integrating several times this equality by parts and using the facts that Π_n and $\mathbb{P}$ are self-adjoint with respect to the L^2-product and that $\Pi_n u_n = u_n$ and $\Pi_n \partial_t u_n = \partial_t u_n$, we obtain, for all $0 \leq t < T_n$,

$$\partial_t E_n + \|\nabla u_n\|_{1+\eta}^2 + \|\partial_t u_n\|_{1+\eta}^2 = I_1 + I_2 + I_3 + I_4, \tag{3.17}$$

where

$$\begin{aligned} I_1 &= \left(J_{1+\eta}(f), J_{1+\eta}(u_n + 2\partial_t u_n)\right), \\ I_2 &= -\left(J_{1+\eta}(u_n.\nabla u_n), J_{1+\eta}(u_n + 2\partial_t u_n)\right), \\ I_3 &= -\left(J_{1+\eta}(\partial_t u_n.\nabla u_n), J_{1+\eta}(u_n + 2\partial_t u_n)\right), \\ I_4 &= -\left(J_{1+\eta}(u_n.\nabla \partial_t u_n), J_{1+\eta}(u_n + 2\partial_t u_n)\right). \end{aligned}$$

The estimate of the term I_1 is immediate. Indeed, applying the Hölder and Young inequalities as well as (3.14), we get

$$\begin{aligned} |I_1(t)| &\leq \|f\|_{1+\eta}\left(\|u_n\|_{1+\eta} + 2\|\partial_t u_n\|_{1+\eta}\right) \\ &\leq 2\|f(t)\|_{1+\eta}^2 + E_n(t) \\ &\leq 2\|f(t)\|_{1+\eta}^2 + 1 + E_n^2(t). \end{aligned} \tag{3.18}$$

To estimate the term I_2, we use the inequality (3.6) of Proposition 3.1 and the fact that $H^{1+\eta}$ is an algebra. By this way, we obtain

$$\begin{aligned} |I_2(t)| &\leq \|u_n.\nabla u_n\|_{1+\eta}\|u_n + 2\partial_t u_n\|_{1+\eta} \\ &\leq C\|u_n\|_{1+\eta}\|\nabla u_n\|_{1+\eta}\|u_n + 2\partial_t u_n\|_{1+\eta}, \end{aligned}$$

from which we deduce

$$|I_2(t)| \leq \frac{1}{4}\|\nabla u_n(t)\|_{1+\eta}^2 + CE_n^2(t). \tag{3.19}$$

The term I_3 is estimated by applying the inequality (3.6) of Proposition 3.1. It comes

$$\begin{aligned} |I_3(t)| &\leq \|\partial_t u_n.\nabla u_n\|_{1+\eta}\|u_n + 2\partial_t u_n\|_{1+\eta} \\ &\leq C\|\partial_t u_n\|_{1+\eta}\|\nabla u_n\|_{1+\eta}\|u_n + 2\partial_t u_n\|_{1+\eta}, \end{aligned}$$

and thus

$$|I_3(t)| \leq \frac{1}{4}\|\nabla u_n(t)\|_{1+\eta}^2 + CE_n^2(t). \tag{3.20}$$

It remains to estimate I_4 that we rewrite as the sum $I_4 = I_{4,1} + I_{4,2}$, where

$$\begin{aligned} I_{4,1} &= -\left(J_{1+\eta}(u_n.\nabla\partial_t u_n), J_{1+\eta}u_n\right), \\ I_{4,2} &= -2\left(J_{1+\eta}(u_n.\nabla\partial_t u_n), J_{1+\eta}\partial_t u_n\right). \end{aligned}$$

Since $\operatorname{div} u_n = 0$, we have $u_n.\nabla\partial_t u_n = \operatorname{div}((u_n \otimes \partial_t u_n)$. Then, an integration by parts yields

$$I_{4,1} = \left(J_{1+\eta}(u_n \otimes \partial_t u_n), J_{1+\eta}\nabla u_n\right).$$

Hence, the inequality (3.6) and a Young inequality imply

$$\begin{aligned} |I_{4,1}(t)| &\leq C\|u_n\|_{1+\eta}\|\partial_t u_n\|_{1+\eta}\|\nabla u_n\|_{1+\eta} \\ &\leq \frac{1}{8}\|u_n(t)\|_{2+\eta}^2 + CE_n^2(t). \end{aligned}$$

In order to bound the term $I_{4,2}$, we first remark that, since u_n is divergence free, the following equality holds:

$$\left(u_n.\nabla J_{1+\eta}\partial_t u_n,\, J_{1+\eta}\partial_t u_n\right) = 0. \tag{3.21}$$

Therefore, $I_{4,2}$ writes

$$I_{4,2} = -2\left([J_{1+\eta}, u_n].\nabla\partial_t u_n,\, J_{1+\eta}\partial_t u_n\right).$$

Then, the commutator inequality (3.10) of Lemma 3.2 and the Cauchy–Schwartz and Young inequalities give

$$\begin{aligned}|I_{4,2}(t)| &\le C\left\|J_{1+\eta}\nabla u_n\right\|_{L^2}\left\|J_\eta\nabla\partial_t u\right\|_{L^2}\|\partial_t u_n\|_{1+\eta}\\ &\le C\|\nabla u_n(t)\|_{1+\eta}\|\partial_t u_n(t)\|_{1+\eta}^2\\ &\le \frac{1}{8}\|\nabla u_n(t)\|_{1+\eta}^2 + CE_n^2.\end{aligned}$$

Finally, adding the estimates of $I_{4,1}$ and $I_{4,2}$, we get

$$|I_4(t)| \le \frac{1}{4}\|\nabla u_n(t)\|_{1+\eta}^2 + CE_n^2(t). \tag{3.22}$$

Finally, going back to the equality (3.17) and taking into account the estimates (3.18), (3.19), (3.20) and (3.22), we conclude that, for $0 \le t < T_n$,

$$\partial_t E_n(t) + \frac{1}{4}\|\nabla u_n(t)\|_{1+\eta}^2 + \|\partial_t u_n(t)\|_{1+\eta}^2 \le 2\|f(t)\|_{1+\eta}^2 + 1 + CE_n^2(t). \tag{3.23}$$

□

The inequality (3.16) of Lemma 3.3 allows to show that there exists a n-independent positive time T, such that the $H^{2+\eta}\times H^{1+\eta}$-norm of $(u_n(t), \partial_t u_n(t))$ is bounded on the time interval $[0, T)$. Indeed, by rewriting the inequality (3.16) as

$$\partial_t\mathcal{E}_n(t) \le G(t)\mathcal{E}_n^2(t), \tag{3.24}$$

where

$$G(t) = C(\|f(t)\|_{1+\eta}^2 + 1) \quad\text{and}\quad \mathcal{E}_n(t) = 1 + E_n(t),$$

we can show the next lemma.

Lemma 3.4 *Let $0 < \eta < 1$ be fixed and let $(u_0, u_1) \in H_\sigma^{2+\eta}\times H_\sigma^{1+\eta}$, $f \in C^0\left([0,+\infty), H_\sigma^{1+\eta}\right)\cap L^\infty\left([0,+\infty), H_\sigma^{1+\eta}\right)$ be given. There exists a time $T > 0$ depending only on $\|u_0\|_{2+\eta}$, $\|u_1\|_{1+\eta}$ and $\|f\|_{L^\infty(H^{1+\eta})}$ such that, for any n, $T < T_n$ and the solution $(u_n, \partial_t u_n) \in C^0\left([0, T_n], \Pi_n(H_\sigma^{2+\eta})\times\Pi_n(H_\sigma^{1+\eta})\right)$ of (3.11) with initial data $(\Pi_n(u_0), \Pi_n(u_1))$ satisfies the following uniform bound with respect to n, for any $t\in[0, T]$,*

$$1 + \frac{1}{8}\left(\|u_n(t)\|_{2+\eta}^2 + \|u_n(t)\|_{1+\eta}^2\right) \le \mathcal{E}_n(t) \le \frac{\mathcal{E}_n(0)}{1 - Ct\mathcal{E}_n(0)\left(1 + \|f\|_{L^\infty([0,+\infty),H^{1+\eta})}^2\right)}. \tag{3.25}$$

Proof The differential inequality (3.24) can be rewritten as follows:

$$-\partial_t\left(\frac{1}{\mathcal{E}_n(t)}\right) \le C\left(1 + \|f(t)\|_{1+\eta}^2\right).$$

The solution $\mathcal{E}_n(t)$ of this differential inequality exists on a positive time interval $[0, \tau_n)$. Integrating the above inequality between 0 and $0 < t < \tau_n$, we obtain,

$$\frac{1}{\mathcal{E}_n(t)} \geq \frac{1}{\mathcal{E}_n(0)} - Ct\left(1 + \|f\|^2_{L^\infty([0,+\infty),H^{1+\eta})}\right).$$

Equivalently, we have

$$\mathcal{E}_n(t) \leq \frac{\mathcal{E}_n(0)}{1 - Ct\,\mathcal{E}_n(0)\left(1 + \|f\|^2_{L^\infty([0,+\infty),H^{1+\eta})}\right)}. \tag{3.26}$$

We next choose $T > 0$ such that

$$0 < T \leq \frac{1}{C\mathcal{E}_n(0)\left(1 + \|f\|^2_{L^\infty([0,+\infty),H^{1+\eta})}\right)}. \tag{3.27}$$

The estimate (3.25) is a direct consequence of (3.26), (3.27) and (3.14). Moreover, due to the inequalities (3.14), we have

$$\mathcal{E}_n(0) \leq 1 + \frac{3}{2}\left(\|u_0\|^2_{2+\eta} + \|u_1\|^2_{1+\eta}\right),$$

which implies that the bound on T in (3.27) as well as the estimate (3.25) are independent of n. □

3.3 Existence of Solutions

We now show the existence of solutions of (2.2), when the initial data (u_0, u_1) belong to the space $H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ and the forcing term f belongs to $C^0\left([0,+\infty), H^{1+\eta}_\sigma\right) \cap L^\infty\left([0,+\infty), H^{1+\eta}_\sigma\right)$, where $0 < \eta < 1$.
The proof of the existence of such solutions is done by using the a priori estimates that we performed in Sect. 3 for smooth initial data and forcing term. More precisely, we consider the regular solution $(u_n, \partial_t u_n) \in C^1\left([0, T_n], \Pi_n(H^{2+\eta}_\sigma) \times \Pi_n(H^{1+\eta}_\sigma)\right)$ of the system (3.11) with initial data $(\Pi_n(u_0), \Pi_n(u_1))$ and forcing term $\Pi_n(f)$. Due to the Lemma 3.4, there exists a positive time T, independent of n, such that $(u_n, \partial_t u_n)$ is bounded in $L^\infty\left([0, T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right)$, uniformly with respect to n. Consequently, up a subsequence of $(u_n, \partial_t u_n)_{n\geq 0}$, there exists $(u, \partial_t u) \in L^\infty\left([0, T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right) \cap C^0_w\left([0, T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right)$ such that

$$\begin{aligned}
&(u_n, \partial_t u_n) \rightharpoonup (u, \partial_t u) \quad \text{weak* in} \quad L^\infty\left([0, T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right),\\
&(u_n(t), \partial_t u_n(t)) \rightharpoonup (u(t), \partial_t u(t)) \quad \text{weakly in} \quad H^{2+\eta}_\sigma \times H^{2+\eta}_\sigma, \quad \text{for all} \quad t \in [0, T].
\end{aligned}$$

The weak continuity of $(u, \partial_t u)$ into $H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ is deduced from the weak convergence of $(u_n(t), \partial_t u_n(t))$ to $(u(t), \partial_t u(t))$, for all $t \in [0, T]$. Let $\varphi \in C^\infty_0(\mathbb{R}^2)$ be a smooth test function and Ω denotes the compact support of φ. Making the L^2-product of (3.11) with φ, we obtain

$$\int_{\Omega}\left(\partial_t^2 u_n+\partial_t u_n-\Delta u_n\right)(t,x)\varphi(x)dx$$
$$+\int_{\Omega}\mathbb{P}\left(u_n.\nabla u_n+\partial_t u_n.\nabla u_n+u_n.\nabla\partial_t u_n\right)(t,x)\Pi_n(\varphi(x))dx$$
$$=int_{\Omega}\mathbb{P}f(t,x)\Pi_n(\varphi(x))dx. \tag{3.28}$$

The idea is to pass to the limit when n goes to infinity and to show that the equality (2.3) holds. Although the linear terms of (3.28) pass easily to the limit, the weak convergence of $(u_n,\partial_t u_n)$ to $(u,\partial_t u)$ is not sufficient to deal with the non-linear terms of (3.28). This is why we need to establish strong convergence between $(u_n,\partial_t u_n)$ and $(u,\partial_t u)$. To do so, we use compactness embedding results of the Sobolev spaces defined on the regular domain Ω. For $s\geq 0$, let $H^s(\Omega)$ denotes the space of the restrictions of the H^s-functions to Ω. In particular, if $s\leq\sigma$, then $H^\sigma(\Omega)$ is compactly embedded in $H^s(\Omega)$. The boundedness properties of u_n and $\partial_t u_n$ in $H^{2+\eta}(\Omega)^2\times H^{1+\eta}(\Omega)^2$ uniformly with respect to n give

$$\begin{aligned} &u_n\rightharpoonup u \quad \text{weak* in} \quad L^\infty\left([0,T],H^{2+\eta}(\Omega)^2\right),\\ &\partial_t u_n\rightharpoonup\partial_t u \quad \text{weak* in} \quad L^\infty\left([0,T],H^{1+\eta}(\Omega)^2\right). \end{aligned} \tag{3.29}$$

In addition, since $(u_n,\partial_t u_n)$ is bounded in $L^\infty\left([0,T],H^{2+\eta}(\Omega)^2\times H^{1+\eta}(\Omega)^2\right)$ uniformly with respect to n, we can check from the first equation of (3.11) that $\partial_t^2 u_n$ is bounded in $L^\infty\left([0,T],L^2(\Omega)^2\right)$, independently from n. Hence, the following convergence results hold:

$$\begin{aligned} &\partial_t^2 u_n\rightharpoonup\partial_t^2 u \quad \text{weak* in} \quad L^\infty\left([0,T],L^2(\Omega)^2\right),\\ &\partial_t^2 u_n(t)\rightharpoonup\partial_t^2 u(t) \quad \text{weakly in} \quad L^2(\Omega)^2, \quad \text{for all} \quad t\in[0,T]. \end{aligned} \tag{3.30}$$

Furthermore, for all $t\in[0,T]$, the set $\bigcup_{n\in\mathbb{N}} u_n(t)$ is compact in $H^{1+\eta}(\Omega)^2$ and u_n is equicontinuous in $H^{1+\eta}(\Omega)^2$. Indeed, for all $t_1,t_2\in[0,T]$, $t_2\geq t_1$, we have

$$\begin{aligned} \|u_n(t_2)-u_n(t_1)\|_{H^{1+\eta}(\Omega)} &\leq \int_{t_1}^{t_2}\|\partial_t u_n(s)\|_{H^{1+\eta}(\Omega)}\,ds\\ &\leq (t_2-t_1)\,\|\partial_t u_n\|_{L^\infty([0,T],H^{1+\eta}(\Omega))}\,ds. \end{aligned}$$

Consequently, the classical Arzela-Ascoli Theorem implies

$$u_n\to u \quad \text{strongly in} \quad C^0\left([0,T],H^{1+\eta}(\Omega)^2\right),$$

which can be improved through interpolation inequalities to

$$u_n\to u \quad \text{strongly in} \quad C^0\left([0,T],H^s(\Omega)^2\right), \quad \text{for all} \quad s\in[0,2+\eta). \tag{3.31}$$

With the same tools, arguing that $\partial_t^2 u_n$ is bounded in $L^\infty\left([0,T],L^2(\Omega)^2\right)$ uniformly with respect to n, we obtain

$$\partial_t u_n\to\partial_t u \quad \text{strongly in} \quad C^0\left([0,T],H^s(\Omega)^2\right), \quad \text{for all} \quad s\in[0,1+\eta). \tag{3.32}$$

Now, the identities (3.31) and (3.32) are sufficient to pass to the limit in the equation (3.28) when n goes to infinity and obtain (2.3) for all $\varphi\in C_0^\infty(\mathbb{R}^2)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense into $L^2(\mathbb{R}^2)$, we conclude that $(u,\partial_t u)$ also satisfies (2.3) for all $\varphi\in L^2(\mathbb{R}^2)$.

3.4 Uniqueness of Solutions

We now show that the solution of (2.2) obtained in Section 3.3 is unique. To this end, we consider two solutions $(u, \partial_t u)$ and $(u^*, \partial_t u^*)$ which belong to $C^0_w([0,T], (H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma)$ with the same initial data $(u_0, u_1) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ and forcing term $f \in C^0\left([0,+\infty), H^{1+\eta}_\sigma\right)$. We define $v = u - u^*$. It is clear that the following equality holds (in the sense of Definition 2.1), for all $t \in [0,T]$,

$$\partial_t^2 v + \partial_t v - \Delta v + u.\nabla v + v.\nabla u^* + \partial_t u.\nabla v + \partial_t v.\nabla u^* + u.\nabla\partial_t v + v.\nabla\partial_t u^* + \nabla q = 0. \tag{3.33}$$

The aim of this subsection is to demonstrate that $v \equiv 0$, which is done by making estimates on the energy functional $\tilde{E}$, given by

$$\tilde{E}(t) = \frac{1}{2}\left(\|v(t) + \partial_t v(t)\|^2 + \|\partial_t v(t)\|^2\right) + \|\nabla v(t)\|^2 .$$

The Definition 2.1 allows to take the L^2-inner product of (3.33) with the L^2-function $v + 2\partial_t v$. Integrating by parts and using the fact that div $u =$ div $u^* = 0$, we obtain, for $0 \le t \le T$,

$$\partial_t \tilde{E} + \|\nabla v\|^2 + \|\partial_t v\|^2 = J_1 + J_2 + J_3 + J_4, \tag{3.34}$$

where

$$\begin{aligned} J_1 &= -2\,(u.\nabla v + \partial_t u.\nabla v, \partial_t v) \\ J_2 &= -\,(v.\nabla u^* + \partial_t v.\nabla u^*, v + 2\partial_t v)\,, \\ J_3 &= -\,(u.\nabla\partial_t v, v)\,, \\ J_4 &= -\,(v.\nabla\partial_t u^*, v + 2\partial_t v)\,. \end{aligned}$$

We now have to estimate each term separately. The estimate of the term J_1 is straightforward. Using a Hölder inequality, we obtain

$$\begin{aligned} |J_1(t)| &\le 2\,\|u(t) + \partial_t u(t)\|_{L^\infty}\,\|\nabla v\|\,\|\partial_t v\| \\ &\le C\,\|u(t) + \partial_t u(t)\|_{L^\infty}\,\tilde{E}(t) \\ &\le C\,\|u + \partial_t u\|_{L^\infty([0,T],L^\infty)}\,\tilde{E}(t) \end{aligned} \tag{3.35}$$

Likewise, we get

$$\begin{aligned} |J_2(t)| &\le \left\|\nabla u^*(t)\right\|_{L^\infty}(\|v(t)\| + 2\,\|\partial_t v(t)\|)^2 \\ &\le C\,\left\|\nabla u^*(t)\right\|_{L^\infty}\left(\|v(t)\|^2 + \|\partial_t v(t)\|^2\right)^2 \\ &\le C\,\left\|\nabla u^*\right\|_{L^\infty([0,T],L^\infty)}\,\tilde{E}(t). \end{aligned} \tag{3.36}$$

Since div $u = 0$, an integration by parts gives

$$J_3 = -\,(\mathrm{div}\,(u \otimes \partial_t v), v) = (u \otimes \partial_t v, \nabla v)\,.$$

Consequently, we have

$$\begin{aligned} |J_3(t)| &\le C\,\|u(t)\|_{L^\infty}\,\|\partial_t v(t)\|\,\|\nabla v(t)\| \\ &\le 2\,\|u(t)\|_{L^\infty}\,\tilde{E}(t) \\ &\le 2\,\|u\|_{L^\infty([0,T],L^\infty)}\,\tilde{E}(t). \end{aligned} \tag{3.37}$$

The last term J_4 is a little more difficult to estimate. We split it into the sum $J_4 = J_{4,1} + J_{4,2}$, where

$$J_{4,1} = -\left(v.\nabla\partial_t u^*, v\right),$$
$$J_{4,2} = -2\left(v.\nabla\partial_t u^*, \partial_t v\right).$$

In order to bound $J_{4,1}$, we apply the following Gagliardo-Nirenberg inequality (also called Ladyzhenskaya inequality)

$$\|v\|_{L^4} \leq C\,\|v\|^{\frac{1}{2}}\,\|\nabla v\|^{\frac{1}{2}}, \tag{3.38}$$

and obtain

$$\begin{aligned}|J_{4,1}(t)| &\leq C\left\|\nabla\partial_t u^*(t)\right\|\,\|v(t)\|_{L^4}^2\\ &\leq C\left\|\nabla\partial_t u^*(t)\right\|\,\|v(t)\|\,\|\nabla v(t)\|\\ &\leq C\left\|\nabla\partial_t u^*\right\|_{L^\infty([0,T],L^2)}\tilde{E}(t).\end{aligned}$$

It remains to estimate $J_{4,2}$. We first recall that $0 < \eta < 1$. Applying the Hölder inequality

$$\|w_1 w_2\|_{L^2} \leq \|w_1\|_{L^{\frac{2}{\eta}}}\,\|w_2\|_{L^{\frac{2}{1-\eta}}}, \ \forall w_1 \in L^{\frac{2}{\eta}}(\mathbb{R}^2),\ \forall w_2 \in L^{\frac{2}{1-\eta}}(\mathbb{R}^2), \tag{3.39}$$

we can write

$$|J_{4,2}(t)| \leq 2\,\|v(t)\|_{L^{\frac{2}{\eta}}}\left\|\nabla\partial_t u^*(t)\right\|_{L^{\frac{2}{1-\eta}}}\|\partial_t v(t)\|.$$

Then, using the continuous injections of $\dot{H}^\eta(\mathbb{R}^2)$ into $L^{\frac{2}{1-\eta}}(\mathbb{R}^2)$ and of $\dot{H}^{1-\eta}(\mathbb{R}^2)$ into $L^{\frac{2}{\eta}}(\mathbb{R}^2)$, we deduce from the above inequality:

$$|J_{4,2}(t)| \leq C\|v(t)\|_{1-\eta}\|\partial_t u^*(t)\|_{1+\eta}\|\partial_t v(t)\| \leq C\|v(t)\|_1\|\partial_t u^*(t)\|_{1+\eta}\|\partial_t v(t)\|,$$

and finally

$$|J_{4,2}(t)| \leq C\|\partial_t u^*\|_{L^\infty(H^{1+\eta})}\tilde{E}(t). \tag{3.40}$$

The equality (3.34) together with the inequalities (3.35), (3.36), (3.37) and (3.40) imply that, for all $t \in [0, T]$,

$$\partial_t\tilde{E}(t) + \|\nabla v(t)\|^2 + \|\partial_t v(t)\|^2 \leq C\tilde{E}(t). \tag{3.41}$$

Since $\tilde{E}(0) = 0$, the Gronwall lemma implies $\tilde{E} \equiv 0$ on the time interval $[0, T]$. Therefore $v(t) = u(t) - u^*(t) = 0$ for all $t \in [0, T]$ and the solutions of (2.2) obtained from Sect. 3.3 are unique.

3.5 Time-Continuity of the Solutions of System (2.2)

Throughout this section, $(u^*, \partial_t u^*)$ denotes a solution of the system (2.2) with initial data $(u_0, u_1) \in H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}$ and forcing term $f \in C^0\left(\mathbb{R}^+, H^{1+\eta}\right)$, with $\eta > 0$. As shown above, $(u^*, \partial_t u^*) \in C_w^0\left([0, T], H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}\right)$. The aim of this section is to show that $(u^*, \partial_t u^*)$ is strongly continuous in time. To do so, we define the rotational of u^*, given by $\omega^* = \partial_1 u_2^* - \partial_2 u_1^*$, and show that $(\omega^*, \partial_t\omega^*) \in C^0\left([0, T], H^{1+\eta} \times H^\eta\right)$. By taking the rotational of the first equation of (2.2) and using the fact that div $u^* = 0$, we can show that ω^* satisfies

$$\partial_t^2\omega^* + \partial_t\omega^* - \Delta\omega^* + u^*.\nabla\omega^* + \partial_t\left(u^*.\nabla\omega^*\right) = \text{curl } f. \tag{3.42}$$

The strategy that we use to establish the time continuity of $(\omega^*, \partial_t \omega^*)$ relies on the decomposition of ω^* as the sum $\omega^* = \omega_1^* + \omega_2^*$, where $(\omega_1^*, \partial_t \omega_1^*)$ is more regular than $(\omega^*, \partial_t \omega^*)$ and continuous in time in the space $H^{1+\eta} \times H^\eta$ and $(\omega_2^*, \partial_t \omega_2^*)$ remains small in $L^\infty\left([0, T], H^{1+\eta} \times H^\eta\right)$. More precisely, we set $(\omega_1^*, \partial_t \omega_1^*)$ to be the solution of the linear Cauchy problem

$$\begin{aligned} &\partial_t^2 \omega_1^* + \partial_t \omega_1^* - \Delta \omega_1^* + u^*.\nabla \omega_1^* + \partial_t \left(u^*.\nabla \omega_1^*\right) = (I - \Pi_m)\,(\text{curl } f), \\ &\omega_{1|t=0}^* = (I - \Pi_m)\,(\text{curl } u_0)\,, \\ &\partial_t \omega_{1|t=0}^* = (I - \Pi_m)\,(\text{curl } u_1)\,, \end{aligned} \tag{3.43}$$

and $(\omega_2^*, \partial_t \omega_2^*)$ to be the solution of the linear Cauchy problem

$$\begin{aligned} &\partial_t^2 \omega_2^* + \partial_t \omega_2^* - \Delta \omega_2^* + u^*.\nabla \omega_2^* + \partial_t \left(u^*.\nabla \omega_2^*\right) = \Pi_m\,(\text{curl } f)\,, \\ &\omega_{2|t=0}^* = \Pi_m\,(\text{curl } u_0)\,, \\ &\partial_t \omega_{2|t=0}^* = \Pi_m\,(\text{curl } u_1)\,, \end{aligned} \tag{3.44}$$

where Π_m is the cut-off operator given by (3.1) and $m \in \mathbb{N}$ will be made more precise later. The study of these two Cauchy problems can be reduced to the study of the following linear one:

$$\begin{aligned} &\partial_t^2 \omega + \partial_t \omega - \Delta \omega + u^*.\nabla \omega + \partial_t\,(u^*.\nabla \omega) = g, \\ &\omega_{j|t=0} = \omega_0, \\ &\partial_t \omega_{j|t=0} = \omega_1, \end{aligned} \tag{3.45}$$

where $(\omega_0, \omega_1) \in H^s \times H^{s-1}$ and $g \in C^0\left(\mathbb{R}^+, H^{s-1}\right)$, with $s \geq 1$. In what follows, we establish the existence of solutions to (3.45) when s belongs to $[1, 2+\eta]$.

3.5.1 Study of the Auxiliary Linear Problem

We now establish the existence and uniqueness of solutions to the Cauchy problem (3.45) when $(\omega_0, \omega_1) \in H^s \times H^{s-1}$ and $g \in C^0\left(\mathbb{R}^+, H^{s-1}\right)$, with $1 \leq s \leq 2+\eta$. The next definition states what is meant by being a solution of (3.45).

Definition 3.1 We call solution of (3.45) with initial data (ω_0, ω_1) a couple $(\omega, \partial_t \omega) \in C_w^0\left([0, T], H^s \times H^{s-1}\right)$ such that, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} (\partial_t \omega(t,x) - \omega_1(x))\,\varphi(x)dx + \int_{\mathbb{R}^2} (\omega(t,x) - \omega_0(x))\,\varphi(x)dx \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \nabla \omega(s,x).\nabla \varphi(x)dxds \\ &= \int_0^t \int_{\mathbb{R}^2} \left(\omega(s,x)u^*(s,x) + \partial_t\left(\omega(s,x)u^*(s,x)\right)\right).\nabla \varphi(x)dxds \\ &\quad + \int_0^t \int_{\mathbb{R}^2} g(s,x)\varphi(x)dxds. \end{aligned} \tag{3.46}$$

The existence of solutions to the problem (3.45) is given by the next lemma.

Lemma 3.5 *Let $s \in [1, 2+\eta]$, $(\omega_0, \omega_1) \in H^s \times H^{s-1}$, $g \in C^0\left([0, T], H^{s-1}\right)$ and $(u^*, \partial_t u^*) \in C_w^0\left([0, T], H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}\right)$. There exists a solution $(\omega, \partial_t \omega)$ to the problem (3.45) such that $(\omega, \partial_t \omega) \in C_w^0\left([0, T], H^s \times H^{s-1}\right)$ and there exists a positive constant C*

such that, for all $t \in [0, T]$,

$$E(\omega)(t) \leq \left(E(0) + \tfrac{1}{2}\|g\|^2_{L^2((0,t),H^{s-1})}\right) \times \exp\left(C\left(\|u^*\|_{L^\infty((0,t),H^{2+\eta})} + \|\partial_t u^*\|_{L^\infty((0,t),H^{1+\eta})}\right)t + 2t\right), \tag{3.47}$$

where

$$E(\omega)(t) = \frac{1}{2}\left(\|\omega(t) + \partial_t\omega(t)\|^2_{H^{s-1}} + \|\partial_t\omega(t)\|^2_{H^{s-1}}\right) + \|\nabla\omega(t)\|^2_{H^{s-1}},$$

and

$$E(0) = \frac{1}{2}\left(\|\omega_0 + \omega_1)\|^2_{H^{s-1}} + \|\omega_1\|^2_{H^{s-1}}\right) + \|\nabla\omega_0\|^2_{H^{s-1}}.$$

Besides, if $s > 1$, then $(\omega, \partial_t\omega) \in C^0\left([0, T], H^\sigma \times H^{\sigma-1}\right)$, for all $\sigma \in [1, s)$.

Proof The proof of this lemma is based on a Friedrichs scheme and a priori estimates. For $n \in \mathbb{N}$, let us consider the regularised problem

$$\begin{aligned} &\partial_t^2\omega_n + \partial_t\omega_n - \Delta\omega_n + \Pi_n\left(u^*.\nabla\omega_n + \partial_t\left(u^*.\nabla\omega_n\right)\right) = \Pi_n(g),\\ &\omega_{n|t=0} = \Pi_n\left(\omega_0\right),\\ &\partial_t\omega_{n|t=0} = \Pi_n\left(\omega_1\right). \end{aligned} \tag{3.48}$$

We can check by the use of the Cauchy-Lipschitz theorem that there exists a unique solution $(\omega_n, \partial_t\omega_n) \in C^1\left([0, T_n], \Pi_n(H^s) \times \Pi_n(H^{s-1})\right)$, where $T_n \in (0, T]$. We now perform the $H^{s-1}-$inner product of the first equation of (3.48) with $\omega_n+2\partial_t\omega_n$. Let $E(\omega_n) : [0, T_n] \to \mathbb{R}$ be the energy functional given by

$$E(\omega_n)(t) = \frac{1}{2}\left(\|\omega_n(t) + \partial_t\omega_n(t)\|^2_{H^{s-1}} + \|\partial_t\omega_n(t)\|^2_{H^{s-1}}\right) + \|\nabla\omega_n(t)\|^2_{H^{s-1}}.$$

Using the fact that $\Pi_n\omega_n = \omega_n$ and $\Pi_n\partial_t\omega_n = \partial_t\omega_n$, we have

$$\begin{aligned} \partial_t E(\omega_n) + \|\partial_t\omega_n\|^2_{H^{s-1}} + \|\nabla\omega_n\|^2_{H^{s-1}} &= (g, \omega_n + 2\partial_t\omega_n)_{H^{s-1}}\\ &- \left(u^*.\nabla\omega_n, \omega_n + 2\partial_t\omega_n\right)_{H^{s-1}} - \left(\partial_t\left(u^*.\nabla\omega_n\right), \omega_n + 2\partial_t\omega_n\right)_{H^{s-1}}. \end{aligned} \tag{3.49}$$

Then, we integrate the previous equality on the time interval $[0, t]$, where $t \in (0, T_n)$, and obtain

$$\begin{aligned} &E(\omega_n)(t) + \|\partial_t\omega_n\|^2_{L^2((0,t),H^{s-1})} + \|\nabla\omega_n\|^2_{L^2((0,t),H^{s-1})}\\ &= E(\omega_n)(0) + L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} L_1 &= \int_0^t (g, \omega_n + 2\partial_t\omega_n)_{H^{s-1}}\, d\sigma,\\ L_2 &= \int_0^t \left(u^*.\nabla\omega_n, \omega_n + 2\partial_t\omega_n\right)_{H^{s-1}}\, d\sigma,\\ L_3 &= \int_0^t \left(u^*.\nabla\partial_t\omega_n, \omega_n + 2\partial_t\omega_n\right)_{H^{s-1}}\, d\sigma,\\ L_4 &= \int_0^t \left(\partial_t u^*.\nabla\omega_n, \omega_n + 2\partial_t\omega_n\right)_{H^{s-1}}\, d\sigma. \end{aligned}$$

We now have to estimate each term L_i, $i = 1, ..., 4$. By the use of Cauchy Schwartz and Young inequalities, it comes

$$\begin{aligned} L_1 &\leq \frac{1}{2} \|g\|^2_{L^2((0,t),H^{s-1})} + \frac{1}{2} \|\omega_n\|^2_{L^2((0,t),H^{s-1})} + \|\partial_t \omega_n\|^2_{L^2((0,t),H^{s-1})} \\ &\leq \frac{1}{2} \|g\|^2_{L^2((0,t),H^{s-1})} + 2\int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned} \tag{3.51}$$

In order to estimate L_2, we need to separate the case $1 < s \leq 2+\eta$ from the case $s = 1$. Let us assume now that $s > 2$. Since u^* is divergence free, then one has $u^*.\nabla\omega_n = \text{div } (\omega_n u^*)$. Consequently, using the inequality (3.6) and Hölder and Young inequalities, we get

$$\begin{aligned} L_2 &\leq \int_0^t \left\|\text{div } (\omega_n u^*)\right\|_{H^{s-1}} \|\omega_n + 2\partial_t \omega_n\|_{H^{s-1}} \, d\sigma \\ &\leq \left\|u^*\right\|_{L^\infty((0,t),H^s)} \|\omega_n\|_{L^2((0,t),H^s)} \|\omega_n + 2\partial_t\omega_n\|_{L^2((0,t),H^{s-1})} \\ &\leq C \left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

with $C > 0$. If $s = 1$, since $H^{2+\eta}(\mathbb{R}^2)$ is continuously embedded into $L^\infty(\mathbb{R}^2)$, then we have

$$\begin{aligned} L_2 &\leq \int_0^t \left\|u^*.\nabla\omega_n\right\| \|\omega_n + 2\partial_t\omega_n\| \, d\sigma \\ &\leq \left\|u^*\right\|_{L^\infty((0,t),L^\infty)} \|\nabla\omega_n\|_{L^2((0,t),L^2)} \|\omega_n + 2\partial_t\omega_n\|_{L^2((0,t),L^2)} \\ &\leq C \left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

Via classical interpolation arguments, we can deduce the case $s \in (1, 2]$ from the cases $s = 1$ and $s > 2$, and obtain, for all $s \in [1, 2+\eta]$,

$$L_2 \leq C \left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma, \tag{3.52}$$

where $C > 0$. The estimate of L_3 is performed by writing $L_3 = L_{3,1} + L_{3,2}$, with

$$\begin{aligned} L_{3,1} &= \int_0^t \left(u^*.\nabla\partial_t\omega_n, \omega_n\right)_{H^{s-1}} d\sigma, \\ L_{3,2} &= 2\int_0^t \left(u^*.\nabla\partial_t\omega_n, \partial_t\omega_n\right)_{H^{s-1}} d\sigma. \end{aligned}$$

We start with $L_{3,1}$ for which the cases $2 < s \leq 2+\eta$ and $1 \leq s \leq 2$ are dealt separately. If $s > 2$, then the inequality (3.6) holds, and an integration by parts gives

$$\begin{aligned} L_{3,1} &= \int_0^t \left(\text{div } (\partial_t\omega_n u^*), \omega_n\right)_{H^{s-1}} d\sigma \\ &= -\int_0^t J_{s-1}(\partial_t\omega_n u^*).\nabla J_{s-1}\omega_n \, d\sigma \\ &\leq C\int_0^t \left\|u^*\right\|_{H^{s-1}} \|\partial_t\omega_n\|_{H^{s-1}} \|\omega_n\|_{H^s} \, d\sigma \\ &\leq C \left\|u^*\right\|_{L^\infty((0,t),H^{s-1})} \|\partial_t\omega_n\|_{L^2((0,t),H^{s-1})} \|\omega_n\|_{L^2((0,t),H^s)} \\ &\leq C \left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

To obtain the estimate for $s \in [1, 2]$, we first consider the case $s = 1$. Performing an integration by parts and using the fact that $H^{2+\eta}(\mathbb{R}^2)$ is continuously embedded into $L^\infty(\mathbb{R}^2)$, it comes

$$\begin{aligned} L_{3,1} &= \int_0^t \text{div } (\partial_t \omega_n u^*)\omega_n \, d\sigma \\ &= -\int_0^t \partial_t \omega_n u^*.\nabla \omega_n \, d\sigma \\ &\leq C \left\| u^* \right\|_{L^\infty((0,t),L^\infty)} \|\partial_t \omega_n\|_{L^2((0,t),L^2)} \|\nabla \omega_n\|_{L^2((0,t),L^2)} \\ &\leq C \left\| u^* \right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

The case $1 < s \leq 2$ is obtained by interpolation arguments between $H^1(\mathbb{R}^2)$ and $H^{2+\eta}(\mathbb{R}^2)$. Likewise, $L_{3,2}$ has to be dealt in two different ways, whether $s > 2$ or not. We start with the case $s > 2$, and rewrite

$$\begin{aligned} L_{3,2} = \int_0^t \left(\left[J_{s-1}, u^* \right] \nabla \partial_t \omega_n, \, J_{s-1} \partial_t \omega_n \right) d\sigma \\ + \int_0^t \left(u^*.\nabla J_{s-1} \partial_t \omega_n, \, J_{s-1} \partial_t \omega_n \right) d\sigma. \end{aligned}$$

Since div $u^* = 0$, an integration by parts shows that

$$\int_0^t \left(u^*.\nabla J_{s-1} \partial_t \omega_n, \, J_{s-1} \partial_t \omega_n \right)_{L^2} d\sigma = 0.$$

Consequently, through the inequality (3.10) we obtain

$$\begin{aligned} L_{3,2} &\leq \int_0^t \left\| J_{s-1} \nabla u^* \right\|_{L^2} \|J_{s-2} \nabla \partial_t \omega_n\|_{L^2} \|J_{s-1} \partial_t \omega_n\|_{L^2} \, d\sigma \\ &\leq C \left\| u^* \right\|_{L^\infty((0,t),H^s)} \|\partial_t \omega_n\|^2_{L^2((0,t),H^{s-1})} \\ &\leq C \left\| u^* \right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

The case $s = 1$ is obvious, since, in this case

$$L_{3,2} = \int_0^t \left(u^*.\nabla \partial_t \omega_n, \, \partial_t \omega_n \right) d\sigma = 0 \leq \left\| u^* \right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma.$$

As we did for $L_{3,1}$, the case $s \in (1, 2]$ is deduced by interpolation. Combining the estimates of $L_{3,1}$ and $L_{3,2}$, we have finally shown

$$L_3 \leq C \left\| u^* \right\|_{L^\infty((0,t),H^{2+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma, \tag{3.53}$$

with $C > 0$. It remains to perform the estimate of L_4. The term L_4 has to be estimated by considering separately the case $s > 2$ and the case $s \in [1, 2]$. If $s \in (2, 2 + \eta]$, the inequality (3.6) implies

$$\begin{aligned} L_4 &\leq \left\| \partial_t u^* \right\|_{L^\infty((0,t),H^{s-1})} \|\nabla \omega_n\|_{L^2((0,t),H^{s-1})} \|\omega_n + 2\partial_t \omega_n\|_{L^2((0,t),H^{s-1})} \\ &\leq C \left\| \partial_t u^* \right\|_{L^\infty((0,t),H^{1+\eta})} \int_0^t E(\omega_n)(\sigma) d\sigma. \end{aligned}$$

If $s = 1$, since the divergence of $\partial_t u^*$ vanishes, an integration by parts shows that $L_4 = \int_0^t \left(\partial_t u^*.\nabla\omega_n, \omega_n\right) d\sigma = 0$. Hence

$$L_4 = 2\int_0^t \left(\partial_t u^*.\nabla\omega_n, \partial_t\omega_n\right) d\sigma.$$

Due to the continuous injection of $H^{1+\eta}(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned} L_4 &\leq \left\|\partial_t u^*\right\|_{L^\infty((0,t),L^\infty)} \|\nabla\omega_n\|_{L^2((0,t),L^2)} \|\partial_t\omega_n\|_{L^2((0,t),L^2)} \\ &\leq C\left\|\partial_t u^*\right\|_{L^\infty((0,t),H^{1+\eta})} \int_0^t E(\omega_n)(\sigma)d\sigma. \end{aligned}$$

Finally, we have shown that, for all $s \in [1, 2+\eta]$, we have

$$L_4 \leq C\left\|\partial_t u^*\right\|_{L^\infty((0,t),H^{1+\eta})} \int_0^t E(\omega_n)(\sigma)d\sigma, \tag{3.54}$$

where $C > 0$.

Now, combining the estimates (3.51), (3.52), (3.53) and (3.54), we deduce that, for all $t \in [0, T_n)$,

$$\begin{aligned} &E(\omega_n)(t) + \|\partial_t\omega_n\|^2_{L^2((0,t),H^{s-1})} + \|\nabla\omega_n\|^2_{L^2((0,t),H^{s-1})} \leq E(\omega_n)(0) + \tfrac{1}{2}\|g\|^2_{L^2((0,t),H^{s-1})} \\ &\quad + \left(C\left(\left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} + \left\|\partial_t u^*\right\|_{L^\infty((0,t),H^{1+\eta})}\right) + 2\right)\int_0^t E(\omega_n)(\sigma)d\sigma, \end{aligned}$$

where C is a positive constant. Finally, the Gronwall inequality implies

$$\begin{aligned} E(\omega_n)(t) \leq &\left(E(\omega_n)(0) + \frac{1}{2}\|g\|^2_{L^2((0,t),H^{s-1})}\right) \\ &\times \exp\left(C\left(\left\|u^*\right\|_{L^\infty((0,t),H^{2+\eta})} + \left\|\partial_t u^*\right\|_{L^\infty((0,t),H^{1+\eta})}\right)t + 2t\right). \end{aligned} \tag{3.55}$$

In particular, this inequality shows that the H^s−norm of w_n and the H^{s-1}−norm of $\partial_t w_n$ remain bounded as long as $t < T$. Consequently, we conclude that $T_n = T$, for all $n \in \mathbb{N}$. Furthermore, this inequality ensures the existence of $(\omega, \partial_t\omega) \in L^\infty_{loc}\left([0,T], H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)\right)$ such that, up to a subsequence,

$$\begin{aligned} &\omega_n \rightharpoonup \omega \quad \text{weak* in} \quad L^\infty_{loc}\left([0,T], H^s(\mathbb{R}^2)\right), \quad \text{when} \quad n \to +\infty, \\ &\partial_t\omega_n \rightharpoonup \partial_t\omega \quad \text{weak* in} \quad L^\infty_{loc}\left([0,T], H^{s-1}(\mathbb{R}^2)\right), \quad \text{when} \quad n \to +\infty. \end{aligned}$$

Then, when passing to the limit when n goes to infinity, the proof that $(\omega, \partial_t\omega)$ satisfies the equality (3.46) is straightforward. Furthermore, by taking the limit when n goes to infinity in the inequality (3.55), we show directly that $(\omega, \partial_t\omega)$ satisfies (3.47). The uniqueness of the solution to (3.45) is easily obtained by noticing that if $(\omega, \partial_t\omega)$ and $(\tilde{\omega}, \partial_t\tilde{\omega})$ are two solutions of (3.45) with the same initial data and forcing term, then $(\omega - \tilde{\omega}, \partial_t\omega - \partial_t\tilde{\omega})$ is also a solution of (3.45) with vanishing initial data and forcing term. Consequently, the inequality (3.47) implies that $(\omega - \tilde{\omega}, \partial_t\omega - \partial_t\tilde{\omega})(t) = (0, 0)$, for all $t \in [0, T]$.

The end of this proof is achieved by showing that, if $s > 1$, then $(\omega, \partial_t\omega) \in C^0\left([0,T], H^\sigma \times H^{\sigma-1}\right)$, for all $1 \leq \sigma < s$. Going back to (3.48), we actually can show that $\partial_t\omega_n(t)$ remains bounded in $H^{-1}(\mathbb{R}^2)$, for all $t \in [0, \tau)$, with $\tau \in [0, T]$, $\tau < +\infty$, uniformly with respect to n and t. Indeed, we have

$$\begin{aligned}\left\|\partial_t^2\omega_n(t)\right\|_{H^{-1}} &\leq \|\partial_t\omega_n(t)\|_{H^{-1}} + \|\Delta\omega_n(t)\|_{H^{-1}} + \|g(t)\|_{H^{-1}} \\ &\quad + \left\|\text{div }(\omega_n u^*)(t)\right\|_{H^{-1}} + \left\|\text{div }(\partial_t(\omega_n u^*))(t)\right\|_{H^{-1}} \\ &\leq \|\partial_t\omega_n(t)\| + \|\omega_n(t)\|_{H^1} + \|g(t)\| \\ &\quad + \left\|(\omega_n u^*)(t)\right\| + \left\|(\partial_t\omega_n u^* + \omega_n\partial_t u^*)(t)\right\|.\end{aligned}$$

Since $u^*(t)$ and $\partial_t u^*(t)$ belong to $L^\infty(\mathbb{R}^2)$ and $(\omega_n(t), \partial_t\omega_n(t))$ is bounded in $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ on $[0, \tau]$, uniformly with respect to n and t, it implies that $\partial_t^2\omega_n(t)$ is bounded in $H^{-1}(\mathbb{R}^2)$ on $[0, \tau]$, uniformly with respect to n and t. Consequently, for all $t_0, t_1 \in [0, \tau]$ such that $t_1 \geq t_0$, we have

$$\begin{aligned}\|\partial_t\omega_n(t_1) - \partial_t\omega_n(t_0)\|_{H^{-1}} &\leq \int_{t_0}^{t_1} \left\|\partial_t^2\omega_n(\sigma)\right\|_{H^{-1}} d\sigma \\ &\leq (t_1 - t_0)\left\|\partial_t^2\omega_n\right\|_{L^\infty((0,\tau),H^{-1})} \leq C(t_1 - t_0),\end{aligned}$$

where $C > 0$ is independent from n. Likewise, we get

$$\|\omega_n(t_1) - \omega_n(t_0)\|_{L^2} \leq \int_{t_0}^{t_1} \|\partial_t\omega_n(\sigma)\|_{L^2}\, d\sigma \leq C(t_1 - t_0).$$

By passing to the limit when n goes to infinity, the same two inequalities occur also for $(\omega, \partial_t\omega)$, which implies that $(\omega, \partial_t\omega)$ is equicontinuous in $L^2(\mathbb{R}^2) \times H^{-1}(\mathbb{R}^2)$. Since $(\omega(t), \partial_t\omega(t))$ is bounded in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, for all $t \in [0, T]$, interpolation inequalities ensure that $(\omega, \partial_t\omega)$ is equicontinuous in $H^\sigma(\mathbb{R}^2) \times H^{\sigma-1}(\mathbb{R}^2)$, for all $1 \leq \sigma < s$. □

3.5.2 End of the Proof of the Time-Continuity of the Solutions of the System (2.2)

We now consider a solution $(u^*, \partial_t u^*) \in L^\infty_{loc}\left([0, T], H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma\right)$ of the hyperbolic Navier–Stokes equations (2.2) with initial data $(u_0, u_1) \in H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$. Let $\tau \in (0, T)$ be a fixed positive time. In order to establish that $(u^*, \partial_t u^*)$ is continuous in time on the interval $[0, \tau)$, we will actually show that $(\omega^*, \partial_t\omega^*) = (\text{curl } u^*, \text{curl } \partial_t u^*)$ belongs to $C^0\left([0, \tau), H^{1+\eta} \times H^\eta\right)$. We first notice that $(\omega^*, \partial_t\omega^*)$ is a solution of (3.45) with initial data $(\text{curl } u_0, \text{curl } u_1) \in H^{1+\eta}_\sigma \times H^\eta_\sigma$ and forcing term $\text{curl } f \in C^0([0, T], H^\eta)$. For some $m \in \mathbb{N}$, let us consider ω_1^* and ω_2^*, the solutions of the Cauchy problems (3.43) and (3.44), respectively. Since (3.43) and (3.44) are linear systems, $(\omega_1^* + \omega_2^*, \partial_t(\omega_1^* + \omega_2^*))$ is a solution of (3.45) with the same initial data and forcing term as ω^*. Since this solution is unique, it implies $\omega^* = \omega_1^* + \omega_2^*$.

Let ε be a positive arbitrary constant and $t_0 \in [0, \tau)$ be a fixed positive time. For all $t \in [0, \tau)$, we have

$$\begin{aligned}&\left\|\omega^*(t) - \omega^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega^*(t) - \partial_t\omega^*(t_0)\right\|_{H^\eta} \\ &\quad\leq \left\|\omega_1^*(t) - \omega_1^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega_1^*(t) - \partial_t\omega_1^*(t_0)\right\|_{H^\eta} \\ &\qquad + \left\|\omega_2^*(t) - \omega_2^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega_2^*(t) - \partial_t\omega_2^*(t_0)\right\|_{H^\eta}.\end{aligned}$$

Due to the inequality (3.47), there exists a constant $C = C(u^*, \partial_t u^*, \tau) > 0$ such that

$$\begin{aligned}&\left\|\omega_1^*(t) - \omega_1^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega_1^*(t) - \partial_t\omega_1^*(t_0)\right\|_{H^\eta}\\&\leq C\Big(\|(I - \Pi_m)\,\mathrm{curl}\ u_0\|_{H^{1+\eta}}\\&\quad + \|(I - \Pi_m)\,\mathrm{curl}\ u_1\|_{H^\eta} + \|(I - \Pi_m)\,\mathrm{curl}\ f\|_{L^2((0,\tau),H^\eta)}\Big).\end{aligned}$$

Consequently, if m is chosen sufficiently large, then we get

$$\left\|\omega_1^*(t) - \omega_1^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega_1^*(t) - \partial_t\omega_1^*(t_0)\right\|_{H^\eta} \leq \frac{\varepsilon}{2}.$$

Besides, since $(\omega_2^*, \partial_t\omega_2^*)$ is a solution to the Cauchy problem (3.45) with the regular initial data $(\Pi_m\mathrm{curl}\ u_0, \Pi_m\mathrm{curl}\ u_1)$ and forcing term $\Pi_m\mathrm{curl}\ f$, then the Lemma 3.5 ensures that $(\omega_2^*, \partial\omega_2^*) \in C^0\left([0,\tau], H^{1+\eta} \times H^\eta\right)$. Consequently, there exists $\gamma > 0$ such that, if $|t - t_0| \leq \gamma$, then

$$\left\|\omega_2^*(t) - \omega_2^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega_2^*(t) - \partial_t\omega_2^*(t_0)\right\|_{H^\eta} \leq \frac{\varepsilon}{2},$$

and finally, if $|t - t_0| \leq \gamma$, then

$$\left\|\omega^*(t) - \omega^*(t_0)\right\|_{H^{1+\eta}} + \left\|\partial_t\omega^*(t) - \partial_t\omega^*(t_0)\right\|_{H^\eta} \leq \varepsilon,$$

which ensures that $(\omega^*, \partial_t\omega^*) \in C^0\left([0,\tau), H^{1+\eta} \times H^\eta\right)$ and consequently $(u^*, \partial_t u^*) \in C^0\left([0,\tau), H^{2+\eta} \times H^{1+\eta}\right)$.

4 Proof of Theorem 2.2: Existence of Global Solutions

This section is devoted to the Proof of Theorem 2.2. More precisely, we show that if the initial data (u_0, u_1) and the forcing term f are small enough, then the solution obtained in Theorem 2.1 is global in positive time. According to the proof of the local existence, it suffices to show that the $H^{2+\eta} \times H^{1+\eta}$-norm of $(u, \partial_t u)$ remains bounded on $[0, T]$. First, we show that, under appropriate smallness assumptions on the initial data, the homogeneous $\dot{H}^{2+\eta} \times \dot{H}^{1+\eta}$-norm of $(u, \partial_t u)$ is bounded. Then, in order to extend this boundedness property to the non-homogeneous Sobolev spaces, we perform energy estimates in the space $H^1 \times L^2$. Notice that, when we go back to the ε-dependent system (1.6), we expect the smallness conditions to disappear when $\varepsilon = 0$. According to the inequalities (2.6) and (2.7), we have to take a special care to the L^2-norms of u_0 and u_1 as well and the $L^1((0, +\infty), L^2)$-norm of the forcing term f, which do not vanish when ε goes to 0. Throughout all this section, we assume that $(u_0, u_1) \in H_\sigma^{2+\eta} \times H_\sigma^{1+\eta}$ and $f \in C^0([0, +\infty), H^{1+\eta})$ satisfy the smallness assumptions of Theorem 2.2.

In what follows, we will actually need a reduced smallness assumption only. More precisely, we assume that the initial data and forcing terms satisfy

$$\begin{aligned}\|u_0\|_{L^\infty}^2 + \|\nabla u_0\|_{L^\infty}^2 + \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^\infty}^2 + \|\nabla u_1\|_{L^2}^2 + \|u_1\|_{\dot{H}^{1+\eta}}^2\\ + \|\Pi_1 f\|_{L_t^2(L^2(\mathbb{R}^2))}^2 + \|(I - \Pi_1) f\|_{L_t^2(\dot{H}^1(\mathbb{R}^2))} \leq \delta,\end{aligned} \tag{4.1}$$

where Π_1 is defined by (3.1) and $\delta > 0$ denotes a fixed constant which is made more precise at the end of this proof. According to Theorem 2.1, there exists a unique local solution $(u, \partial_t u) \in C^0([0, T), H_\sigma^{2+\eta} \times H_\sigma^{1+\eta})$ of (2.2) with initial data (u_0, u_1), where $T > 0$ is the maximal time of existence of the solution. Moreover, due to the continuity of $(u, \partial_t u)$, there exists a positive time $\tau \in (0, T]$, such that, for all $t \in (0, \tau]$,

$$A(t) \equiv \|u(t)\|^2_{L^\infty} + \|\nabla u(t)\|^2_{L^\infty} + \|\nabla u(t)\|^2_{L^2} + \|\partial_t u(t)\|^2_{\dot{H}^{1+\eta}} + \|\partial_t u(t)\|^2_{L^\infty} + \|\nabla \partial_t u(t)\|^2_{L^2} < 2\delta \,. \tag{4.2}$$

In particular, if τ is finite and $t = \tau$, we have

$$A(\tau) = \|u(\tau)\|^2_{L^\infty} + \|\nabla u(\tau)\|^2_{L^\infty} + \|\nabla u(\tau)\|^2_{L^2} + \|\partial_t u(\tau)\|^2_{\dot{H}^{1+\eta}} + \|\partial_t u(\tau)\|^2_{L^\infty} + \|\nabla \partial_t u(\tau)\|^2_{L^2} = 2\delta \,. \tag{4.3}$$

The proof of the boundedness property of $(u, \partial_t u)$ in $H^{2+\eta} \times H^{1+\eta}$ mostly relies on a decomposition of u into low and high frequencies parts. We decompose u into the sum

$$u = v + w \,,$$

where $v = \Pi_1 u$ and $w = (I - \Pi_1)u$. We can easily check that, for all $t \in [0, T]$, v and w satisfy the following equalities, in the sense of Theorem 2.1:

$$\partial_t^2 v + \partial_t v - \Delta v + \Pi_1 \left(u.\nabla u + \partial_t(u.\nabla u)\right) + \nabla q = \Pi_1(f) \,, \tag{4.4}$$

and

$$\partial_t^2 w + \partial_t w - \Delta w + (I - \Pi_1) \left(u.\nabla u + \partial_t(u.\nabla u)\right) + \nabla r = (I - \Pi_1)(f) \,. \tag{4.5}$$

In what follows, we will perform energy estimates on v and w separately, assuming that the smallness condition (4.2) holds. Notably, we take advantage of two facts. First, v is as regular as wanted and, due to Lemma 3.1, we only have to exhibit bounds on less regular spaces. Indeed, we will show that $(v, \partial_t v)$ is bounded in $H^2 \times H^1$ only. Secondly, we also take advantage of the fact that $(w, \partial_t w)$ satisfies the Poincaré type inequalities (3.3) of Lemma 3.1. In fact, we only have to perform estimates in the Homogeneous spaces $\dot{H}^{2+\eta} \times \dot{H}^{1+\eta}$.

Remark 4.1 If we had to justify these computations rigorously, we should introduce another Friedrichs mollifier Π_n, $n \geq 1$, and perform a priori estimates on $(\Pi_n v, \Pi_n \partial_t v)$ and $(\Pi_n w, \Pi_n \partial_t w)$. Afterwards, as in Sect. 3.2, we would let n go to $+\infty$ and conclude that these boundedness properties also hold for the limit system. In order to slightly shorten and simplify the demonstration, we prefer doing these computations formally. Notice that all the estimates made on $(w, \partial_t w)$ are actually performed on $(\Pi_n w, \Pi_n \partial_t w)$, which is more regular. Consequently, some intermediate terms can involve higher regularity than only $H^{2+\eta} \times H^{1+\eta}$, but the final result does not.

4.1 Estimates in $\dot{H}^{2+\eta}(\mathbb{R}^2)^2 \times \dot{H}^{1+\eta}(\mathbb{R}^2)^2$

In this subsection, we perform estimates on $(v, \partial_t v)$ and $(w, \partial_t w)$ in homogeneous Sobolev spaces. Notice that since $w = (I - \Pi_1)u$, it also gives estimates on the non-homogeneous norms of $(w, \partial_t w)$.

4.1.1 Low Frequencies

Here we want to estimate the term $(\nabla v, \nabla \partial_t v)$ in the $H^{1+\eta} \times H^\eta$-norm. Due to the inequality (3.2) of the Lemma 3.1, it actually reduces to obtain estimates in the $H^1 \times L^2$-norm. Let E_v

be the energy functional defined by

$$E_v(t) = \frac{1}{2}\left(\|\nabla v(t) + \nabla\partial_t v(t)\|^2 + \|\nabla\partial_t v(t)\|^2\right) + \|\Delta v(t)\|^2 .$$

The energy estimates of $(v, \partial_t v)$ is given through the next lemma.

Lemma 4.1 *Let $0 < \eta < 1$ and (u_0, u_1) and f satisfy the assumptions of Theorem 2.1. Assume that the smallness assumption (4.2) is satisfied. Then, there exists $\delta_0 > 0$ such that, if $\delta < \delta_0$, then, for all $t \in [0, \tau]$, the following energy estimate holds.*

$$\begin{aligned} &E_v(t) + \frac{1}{8}\int_0^t \|\Delta v(s)\|^2\, ds + \frac{1}{8}\int_0^t \|\nabla\partial_t v(s)\|^2\, ds \\ &\quad\le E_v(0) + 4\|\Pi_1 f\|^2_{L^2([0,t],L^2)} + C\delta\left(\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})}\right), \end{aligned} \tag{4.6}$$

with $C > 0$.

Proof Taking the L^2-inner product of (4.4) with $-\Delta\,(v - 2\partial_t v)$ and integrating in time from 0 to $t \in [0, \tau]$, we obtain

$$E_v(t) + \int_0^t \|\Delta v(s)\|^2\, \mathrm{d}s + \int_0^t \|\nabla\partial_t v(s)\|^2\, \mathrm{d}s = E_v(0) + I + II + III\ , \tag{4.7}$$

where

$$\begin{aligned} I &= \int_0^t \big(u.\nabla u,\, \Delta(v + 2\partial_t v)\big)(s)\, \mathrm{d}s, \\ II &= \int_0^t \big(\partial_t(u.\nabla u),\, \Delta(v + 2\partial_t v)\big)(s)\, \mathrm{d}s, \\ III &= \int_0^t \big(\Pi_1(f),\, -\Delta(v + 2\partial_t v)\big)(s)\, \mathrm{d}s, \end{aligned}$$

The estimate of the term III is straightforward. Indeed, applying Lemma 3.1 and the Hölder inequality, we obtain

$$|III(t)| \le \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + \frac{1}{8}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \frac{7}{2}\|\Pi_1 f\|^2_{L^2([0,t],L^2)} . \tag{4.8}$$

We now estimate the term I . Using the fact that u, v and w are divergence free and that $(v.\nabla v, \Delta v)_{L^2} = 0$, we can rewrite I as follows:

$$\begin{aligned} I(t) = &\int_0^t (v.\nabla w,\, \Delta v)\,(s)\, \mathrm{d}s + \int_0^t (w.\nabla(v + w),\, \Delta v)\,(s)\, \mathrm{d}s \\ &+ 2\int_0^t (u.\nabla v,\, \Delta\partial_t v)\,(s)\, \mathrm{d}s + 2\int_0^t (u.\nabla w,\, \Delta\partial_t v)\,(s)\, \mathrm{d}s \\ \equiv &I_1 + I_2 + I_3 + I_4\ . \end{aligned}$$

When dealing with I_j, $j = 1, 2, 3, 4$, we mainly use the Hölder and Young inequalities, the classical Sobolev embedding properties and Gagliardo-Nirenberg inequalities in $\mathbb{R}^2$. In particular, we make use of the previously introduced Ladyzhenskaya inequality (3.38). The estimate of I_1 is straighforward. Indeed, applying Lemma 3.1, we get

$$\begin{aligned}|I_1(t)| \leq& \int_0^t \|v\|_{L^\infty}\|\nabla w\|\,\|\Delta v\|\,\mathrm{d}s\\ \leq& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + 2\|v\|^2_{L^\infty([0,t],L^\infty)}\|\nabla w\|^2_{L^2([0,t],L^2)}\\ \leq& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + C\|v\|^2_{L^\infty([0,t],L^\infty)}\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})},\end{aligned} \tag{4.9}$$

where C is a positive constant which may change from one line to another. Let us estimate the term I_2. Using the classical Sobolev estimates and applying Lemma 3.1, we obtain

$$\begin{aligned}|I_2(t)| \leq& \int_0^t \|w\|_{L^\infty}(\|\nabla v\| + \|\nabla w\|)\|\Delta v\|\,\mathrm{d}s\\ \leq& \left(\int_0^t 2\|w\|^2_{L^\infty}(\|\nabla v\|^2 + \|\nabla w\|^2)\,\mathrm{d}s\right)^{1/2}\|\Delta v\|_{L^2([0,t],L^2)}\\ \leq& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + 8\big(\|\nabla v\|^2_{L^\infty([0,t],L^2)} + \|\nabla w\|^2_{L^\infty([0,t],L^2)}\big)\|w\|^2_{L^2([0,t],L^\infty)}\\ \leq& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + C\big(\|\nabla v\|^2_{L^\infty([0,t],L^2)} + \|\nabla w\|^2_{L^\infty([0,t],L^2)}\big)\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})}.\end{aligned} \tag{4.10}$$

An integration by part yields

$$\begin{aligned}I_3(t) =& -2\int_0^t\int_{\mathbb{R}^2} \nabla u\nabla v\nabla\partial_t v\,\mathrm{d}x\,\mathrm{d}s - 2\int_0^t\int_{\mathbb{R}^2} u\nabla^2 v\nabla\partial_t v\,\mathrm{d}x\,\mathrm{d}s\\ \leq& 2\int_0^t \|\nabla u\|_{L^4}\|\nabla v\|_{L^4}\|\nabla\partial_t v\|\,\mathrm{d}s + 2\int_0^t \|u\|_{L^\infty}\|\Delta v\|\|\nabla\partial_t v\|\,\mathrm{d}s.\end{aligned} \tag{4.11}$$

Next, applying the inequality (3.38) to (4.11), as well as a Young inequality and Lemma 3.1, we obtain

$$\begin{aligned}|I_3(t)| \leq& \frac{1}{8}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + 16\int_0^t \|u\|^2_{L^\infty([0,t],L^\infty)}\|\Delta v\|^2_{L^2([0,t],L^2)}dt\\ &+ C\int_0^t \|\nabla u\|\|\nabla v\|\|\Delta v\|\big(\|\Delta v\| + \|\Delta w\|\big)\,\mathrm{d}s\\ \leq& \frac{1}{8}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + C\|\nabla u\|_{L^\infty([0,t],L^2)}\|\nabla v\|_{L^\infty([0,t],L^2)}\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})}\\ &+ \Big(16\|u\|^2_{L^\infty([0,t],L^\infty)} + C\|\nabla u\|_{L^\infty([0,t],L^2)}\|\nabla v\|_{L^\infty([0,t],L^2)}\Big)\|\Delta v\|^2_{L^2([0,t],L^2)}.\end{aligned} \tag{4.12}$$

The estimate of the term I_4 is a simple consequence of the Hölder and Young inequalities and of Lemma 3.1. It comes

$$\begin{aligned}|I_4(t)| \leq& 2\int_0^t \|u\|_{L^\infty}\|\nabla w\|\|\Delta\partial_t v\|\,\mathrm{d}s\\ \leq& C\int_0^t \|u\|_{L^\infty}\|w\|_{\dot{H}^{2+\eta}}\|\nabla\partial_t v\|\,\mathrm{d}s\\ \leq& \frac{1}{8}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + C\|u\|^2_{L^\infty([0,t],L^\infty)}\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})}.\end{aligned} \tag{4.13}$$

It remains to estimate the term II. We first remind that $(v.\nabla v, \Delta v)_{L^2} = 0$, which implies that $\partial_t((v.\nabla v, \Delta v)_{L^2}) = 0$ and

$$\int_0^t (\partial_t v.\nabla v, \Delta v)(s)\,\mathrm{d}s = -\int_0^t (v.\nabla v, \Delta\partial_t v)(s)\,\mathrm{d}s - \int_0^t (v.\nabla\partial_t v, \Delta v)(s)\,\mathrm{d}s\ .$$

The above remark allows us to write $II = II_0 + II_1 + II_2 + II_3 + II_4$, where,

$$\begin{aligned}
II_0 &= -\int_0^t (v.\nabla v, \Delta\partial_t v)\,(s)\,\mathrm{d}s\\
II_1 &= \int_0^t (\partial_t u.\nabla w, \Delta v)_{L^2}\,(s)\,\mathrm{d}s + \int_0^t (\partial_t w.\nabla v, \Delta v)_{L^2}\,(s)\,\mathrm{d}s\\
II_2 &= 2\int_0^t (\partial_t u.\nabla u, \Delta\partial_t v)_{L^2}\,(s)\,\mathrm{d}s,\\
II_3 &= \int_0^t (u.\nabla\partial_t u, \Delta v)_{L^2}\,(s)\,\mathrm{d}s - \int_0^t (v.\nabla\partial_t v, \Delta v)_{L^2}\,(s)\,\mathrm{d}s,\\
II_4 &= 2\int_0^t (u.\nabla\partial_t u, \Delta\partial_t v)_{L^2}\,(s)\,\mathrm{d}s.
\end{aligned}$$

The term II_0 is estimated as the term I_3. Performing an integration by parts as in (4.11) and arguing as in (4.12), we get

$$\begin{aligned}
|II_0(t)| \le \frac{1}{8}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)}\\
+4\left(\|v\|^2_{L^\infty([0,t],L^\infty)} + C\|\nabla v\|^2_{L^\infty([0,t],L^2)}\right)\|\Delta v\|^2_{L^2([0,t],L^2)}
\end{aligned} \tag{4.14}$$

Besides, the Lemma 3.1 implies

$$\begin{aligned}
|II_1(t)| \le& C\int_0^t \|\partial_t u\|_{L^\infty}\|w\|_{\dot H^{2+\eta}}\|\Delta v\|\,\mathrm{d}s + C\int_0^t \|\partial_t w\|_{\dot H^{1+\eta}}\|\nabla v\|_{L^\infty}\|\Delta v\|\,\mathrm{d}s\\
\le& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + C\|\nabla v\|^2_{L^\infty([0,t],L^2)}\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\\
&+ C\|\partial_t u\|^2_{L^\infty([0,t],L^\infty)}\|w\|^2_{L^2([0,t],\dot H^{2+\eta})}.
\end{aligned} \tag{4.15}$$

The estimate of II_2 is done by writing

$$II_2(t) = 2\int_0^t (\partial_t u.(\nabla v + \nabla w), \Delta\partial_t v)_{L^2}\,(s)\,\mathrm{d}s.$$

Then, we integrate by parts the term containing ∇v as in I_3 (see (4.11)). This allows us to decompose II_2 into $II_2 = II_{2,1} + II_{2,2} + II_{2,3}$, with

$$\begin{aligned}
II_{2,1} =& -2\int_0^t (\nabla\partial_t u.\nabla v, \nabla\partial_t v)\,(s)\,\mathrm{d}s, \quad II_{2,2} = -2\int_0^t \left(\partial_t u.\nabla^2 v, \nabla\partial_t v\right)(s)\,\mathrm{d}s,\\
II_{2,3} =& 2\int_0^t (\partial_t u.\nabla w, \Delta\partial_t v)\,(s)\,\mathrm{d}s.
\end{aligned}$$

In order to estimate $II_{2,1}$, we make use of the classical Agmon inequality, which establishes that, for all $z \in H^2(\mathbb{R}^2)$,

$$\|z\|^2_{L^\infty} \le C\|z\|_{L^2}\|\Delta z\|_{L^2}\ , \tag{4.16}$$

where C is a positive constant. Applying Lemma 3.1 several times as well as (4.16), we get

$$\begin{aligned}|II_{2,1}(t)| &\leq 2\int_0^t \|\nabla\partial_t v + \nabla\partial_t w\|\|\nabla v\|_{L^\infty}\|\nabla\partial_t v\|\,\mathrm{d}s \\ &\leq \frac{1}{16}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + 8\|\nabla v\|^2_{L^\infty([0,t],L^\infty)} \\ &\quad\times\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\nabla\partial_t w\|^2_{L^2([0,t],L^2)}\big) \\ &\leq \frac{1}{16}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + C\|\nabla v\|^2_{L^\infty([0,t],L^2)} \\ &\quad\times\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})}\big).\end{aligned} \tag{4.17}$$

The estimates of the terms $II_{2,2}$ and $II_{2,3}$ are very similar and obtained through Lemma 3.1. We thus obtain

$$\begin{aligned}|II_{2,2}(t)| \leq& 2\int_0^t \|\partial_t u\|_{L^\infty}\|\nabla^2 v\|\|\nabla\partial_t v\|\,\mathrm{d}s \\ \leq& \frac{1}{16}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + 16\|\partial_t u\|^2_{L^\infty([0,t],L^\infty)}\|\Delta v\|^2_{L^2([0,t],L^2)},\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}|II_{2,3}(t)| \leq& 2\int_0^t \|\partial_t u\|_{L^\infty}\|\nabla w\|\|\Delta\partial_t v\|\,\mathrm{d}s \\ \leq& C\int_0^t \|\partial_t u\|_{L^\infty}\|w\|_{\dot{H}^{2+\eta}}\|\nabla\partial_t v\|\,\mathrm{d}s \\ \leq& \frac{1}{16}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + C\|\partial_t u\|^2_{L^\infty([0,t],L^\infty)}\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})}.\end{aligned} \tag{4.19}$$

The estimates of II_3 and II_4 are direct consequences of the Hölder and Young inequalities and (3.3). Indeed, we have

$$\begin{aligned}|II_3(t)| \leq& \int_0^t \|u\|_{L^\infty}\|\nabla\partial_t w\|\|\Delta v\|\,\mathrm{d}s + \int_0^t \|w\|_{L^\infty}\|\nabla\partial_t v\|\|\Delta v\|\,\mathrm{d}s \\ \leq& \frac{1}{8}\|\Delta v\|^2_{L^2([0,t],L^2)} + C\|u\|^2_{L^\infty([0,t],L^\infty)}\|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})} \\ &+ 4\|w\|^2_{L^\infty([0,t],L^\infty)}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)},\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}|II_4(t)| \leq& 2\int_0^t \|u\|_{L^\infty}\left(\|\nabla\partial_t v\| + \|\nabla\partial_t w\|\right)\|\Delta\partial_t v\|\,\mathrm{d}s \\ \leq& \frac{1}{16}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} \\ &+ C\|u\|^2_{L^\infty([0,t],L^\infty)}\left(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})}\right).\end{aligned} \tag{4.21}$$

Finally, assuming that $t \leq \tau$ and going back to (4.7), we can combine the estimates (4.8) of III with the estimates (4.9)–(4.13) of I and the estimates (4.14)–(4.21) of II, and obtain via the smallness assumption (4.2)

$$
\begin{aligned}
&E_v(t) + \frac{1}{4}\int_0^t \|\Delta v(s)\|^2\,\mathrm{d}s + \frac{1}{4}\int_0^t \|\nabla\partial_t v(s)\|^2\,\mathrm{d}s \\
&\quad \le E_v(0) + 4\|\Pi_1 f\|^2_{L^2([0,t],L^2)} \\
&\qquad + C\delta\Big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})}\Big),
\end{aligned}
$$

where C is a positive constant independent from u. Taking δ sufficiently small so that $C\delta < 1/8$ gives the inequality (4.6) and achieves the proof of this lemma. □

4.1.2 High Frequencies

Since the vector $(w, \partial_t w)$ does not contain the lowest frequencies terms, Lemma 3.1 ensures that estimates on the homogeneous $(\dot{H}^{2+\eta} \times \dot{H}^{1+\eta})$-norm of $(w, \partial_t w)$ also provides estimates in the non-homogeneous Sobolev spaces $H^{2+\eta} \times H^{1+\eta}$. For $s > 0$, we recall the $\dot{H}^s$ inner-product definition and its associated norm, given by

$$
(u, v)_{\dot{H}^s} = \left(\Lambda^s u, \Lambda^s v\right), \quad \text{and } \|u\|_{\dot{H}^s} = \|\Lambda^s u\|,
$$

where Λ^s is the operator

$$
\Lambda^s u = \mathcal{F}^{-1}\left(|\xi|^s\,\hat{u}\right), \text{ for } u \in \dot{H}^s(\mathbb{R}^2).
$$

We emphasise that Λ^s also satisfies the commutator inequality (3.10) in Lemma 3.2, that is, for any $s > 1$, there exists a constant $C = C_s > 0$ such that, for any $u \in H^{s+1}(\mathbb{R}^2)$ and $v \in H^{s-1}(\mathbb{R}^2)$, we have

$$
\left\|\left[\Lambda^s, u\right] v\right\|_{L^2} \le C\left\|\Lambda^s \nabla u\right\|_{L^2}\left\|\Lambda^{s-1} v\right\|_{L^2}. \tag{4.22}
$$

Let E_w be the energy functional defined by

$$
E_w(t) = \frac{1}{2}\left(\|w + \partial_t w\|^2_{\dot{H}^{1+\eta}} + \|\partial_t w\|^2_{\dot{H}^{1+\eta}}\right) + \|w\|^2_{\dot{H}^{2+\eta}}.
$$

The energy estimate of $(w, \partial_t w)$ in $H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$ is given by the following lemma.

Lemma 4.2 *Let $0 < \eta < 1$ and (u_0, u_1) and f satisfy the assumptions of Theorem 2.1. Assume that the smallness assumption (4.2) is satisfied. Then, there exists $\delta_0 > 0$ such that, if $\delta < \delta_0$, then, for all $t \in [0, \tau]$, the following energy estimate holds:*

$$
\begin{aligned}
E_w(t) + \frac{1}{4}\big(\|w\|^2_{L^2([0,t],\dot{H}^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot{H}^{1+\eta})}\big) \\
\le E_w(0) + C\|(I - \Pi_1) f\|^2_{L^2([0,t],\dot{H}^{1+\eta})} \\
+ \frac{1}{16}\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)}\big),
\end{aligned} \tag{4.23}
$$

where C is a positive constant.

Proof Taking the $\dot{H}^{1+\eta}$-inner product of (4.5) with $w + 2\partial_t w$ and integrating in time from 0 to $t \in [0, \tau]$, we obtain

$$
\begin{aligned}
&E_w(t) + \int_0^t \|w(s)\|^2_{\dot{H}^{2+\eta}}\,\mathrm{d}s + \int_0^t \|\partial_t w(s)\|^2_{\dot{H}^{1+\eta}}\,\mathrm{d}s \\
&\quad = E_w(0) + I^* + II^* + III^* + IIII^*,
\end{aligned} \tag{4.24}
$$

where

$$I^* = \int_0^t \big(u.\nabla u,\, w + 2\partial_t w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s,$$

$$II^* = \int_0^t \big(\partial_t u \nabla u,\, w + 2\partial_t w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s,$$

$$III^* = \int_0^t \big(u \nabla \partial_t u,\, w + 2\partial_t w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s,$$

$$IIII^* = \int_0^t \big((I - \Pi_1) f,\, w + 2\partial_t w\big)_{\dot H^{1+\eta}}$$

Applying Lemma 3.1 and a Young inequality, we immediately get

$$|IIII^*(t)| \le \frac{1}{12}(\|w\|^2_{L^2((0,t),\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2((0,t),\dot H^{1+\eta})}) + C\|(I - \Pi_1) f\|^2_{L^2((0,t),\dot H^{1+\eta})}\,. \tag{4.25}$$

We next estimate the term I^*, that we decompose into the sum $I^* = I_1^* + I_2^*$, where

$$I_1^* = \int_0^t \big(u.\nabla u,\, w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s, \quad I_2^* = 2\int_0^t \big(u.\nabla u,\, \partial_t w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s.$$

Applying the inequality (3.9) and Lemma 3.1, we obtain

$$\begin{aligned}|I_1^*(t)| \le& C\int_0^t \|u\|_{L^\infty}\|u\|_{\dot H^{2+\eta}}\|w\|_{\dot H^{1+\eta}}\, \mathrm{d}s\\ \le& C\int_0^t \|u\|_{L^\infty}\big(\|v\|_{\dot H^{2+\eta}} + \|w\|_{\dot H^{2+\eta}}\big)\|w\|_{\dot H^{1+\eta}}\, \mathrm{d}s\\ \le& C\int_0^t \|u\|_{L^\infty}\big(\|\Delta v\|_{L^2}\|w\|_{\dot H^{2+\eta}} + \|w\|^2_{\dot H^{2+\eta}}\big) ds\ ,\end{aligned} \tag{4.26}$$

Applying several times the Young inequality and using the smallness hypotheses (4.2) and (4.3), we obtain, for $0 \le t \le \tau$,

$$\begin{aligned}|I_1^*(t)| \le& \frac{1}{12}\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + C\|u\|^2_{L^\infty([0,t],L^\infty)}\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big)\\ \le& (\frac{1}{12} + C\delta)\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + C\delta\|\Delta v\|^2_{L^2([0,t],L^2)}\,.\end{aligned} \tag{4.27}$$

Arguing as in the above estimates, by applying the inequality (3.9), Lemma 3.1, the Young inequality, it comes

$$\begin{aligned}|I_2^*(t)| \le& C\int_0^t \|u\|_{L^\infty}\big(\|\Delta v\|_{L^2} + \|w\|_{\dot H^{2+\eta}}\big)\|\partial_t w\|_{\dot H^{1+\eta}} ds\\ \le& \frac{1}{12}\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + C\|u\|^2_{L^\infty([0,t],L^\infty)}\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big).\end{aligned}$$

Then, using the smallness hypotheses (4.2) and (4.3), we deduce from the above inequality that, for $0 \le t \le \tau$,

$$|I_2^*(t)| \le \frac{1}{12}\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + C\delta\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big)\,. \tag{4.28}$$

Thus, we infer from the inequalities (4.27) and (4.28), that, for $0 \le t \le \tau$,

$$|I^*(t)| \le \Big(\frac{1}{12} + C\delta\Big) \|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \frac{1}{12} \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + C\delta \|\Delta v\|^2_{L^2([0,t],L^2)}. \tag{4.29}$$

Let us now deal with II^*, that we decompose into the sum $II^* = II_1^* + II_2^*$, where

$$II_1^* = \int_0^t \big(\partial_t u.\nabla u, w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s\ , \quad II_2^* = 2\int_0^t \big(\partial_t u.\nabla u, \partial_t w\big)_{\dot H^{1+\eta}}\, \mathrm{d}s.$$

Applying the inequality (3.5) and Lemma 3.1, we obtain

$$\begin{aligned} |II_1^*(t)| \le& C\int_0^t \big(\|\partial_t u\|_{L^\infty}\|\nabla u\|_{\dot H^{1+\eta}} + \|\nabla u\|_{L^\infty}\|\partial_t u\|_{\dot H^{1+\eta}}\big)\|w\|_{\dot H^{1+\eta}}\, \mathrm{d}s \\ \le& C\int_0^t \Big[\|\partial_t u\|_{L^\infty}\big(\|w\|_{\dot H^{2+\eta}} + \|\Delta v\|\big) \\ &+ \|\nabla u\|_{L^\infty}\big(\|\partial_t \nabla v\| + \|\partial_t w\|_{\dot H^{1+\eta}}\big)\Big]\|w\|_{\dot H^{2+\eta}}\, \mathrm{d}s\ , \end{aligned} \tag{4.30}$$

Again, applying Young inequalities and using the smallness hypotheses (4.2) and (4.3), we deduce from (4.30) that, for $0 \le t \le \tau$,

$$\begin{aligned} |II_1^*(t)| \le& \frac{1}{12}\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} \\ &+ C\|\nabla u\|^2_{L^\infty([0,t],L^\infty)}\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big) \\ &+ C\|\partial_t u\|^2_{L^\infty([0,t],L^\infty)}\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) \\ \le& \frac{1}{12}\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} \\ &+ C\delta\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)} \\ &\quad + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big)\ . \end{aligned} \tag{4.31}$$

Likewise, the term II_2^* is dealt by using the same arguments, leading to

$$\begin{aligned} |II_2^*(t)| \le& C\int_0^t \Big[\|\partial_t u\|_{L^\infty}\big(\|w\|_{\dot H^{2+\eta}} + \|\Delta v\|\big) \\ &+ \|\nabla u\|_{L^\infty}\big(\|\partial_t \nabla v\| + \|\partial_t w\|_{\dot H^{1+\eta}}\big)\Big]\|\partial_t w\|_{\dot H^{1+\eta}}\, \mathrm{d}s\ . \end{aligned} \tag{4.32}$$

Again, using the smallness hypotheses (4.2) and (4.3), we derive from (4.32) that, for $0 \le t \le \tau$,

$$\begin{aligned} |II_2^*(t)| \le& \frac{1}{12}\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} \\ &+ C\delta\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)} \\ &\quad + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big)\ . \end{aligned} \tag{4.33}$$

Adding the estimates (4.31) and (4.33), we get, for $0 \le t \le \tau$,

$$
\begin{aligned}
|II^*(t)| \le & \frac{1}{12}\big(\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) \\
& + C\delta\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)} \\
& \qquad + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big)\,.
\end{aligned}
\tag{4.34}
$$

The only remaining term is III^*, that we decompose as follows:

$$
\begin{aligned}
III^* = & \int_0^t \big(u\nabla\partial_t v,\, w + 2\partial_t w\big)_{\dot H^{1+\eta}}\,\mathrm{d}s + \int_0^t \big(u\nabla\partial_t w,\, w\big)_{\dot H^{1+\eta}}\,\mathrm{d}s \\
& + 2\int_0^t \big(u\nabla\partial_t w,\, \partial_t w\big)_{\dot H^{1+\eta}}\,\mathrm{d}s \equiv III_1^* + III_2^* + III_3^*\,.
\end{aligned}
$$

We first estimate the term III_1^*. Arguing as above, by applying the inequality (3.9) as well as Lemma 3.1, we obtain

$$
\begin{aligned}
|III_1^*(t)| \le & C\int_0^t \big(\|u\|_{L^\infty}\|\partial_t v\|_{\dot H^{2+\eta}} + \|\partial_t v\|_{L^\infty}\|u\|_{\dot H^{2+\eta}}\big)\|w + 2\partial_t w\|_{\dot H^{1+\eta}}\,\mathrm{d}s \\
\le & C\int_0^t \Big(\|u\|_{L^\infty}\|\nabla\partial_t v\| + \|\partial_t v\|_{L^\infty}\big(\|\Delta v\| + \|w\|_{\dot H^{2+\eta}}\big)\Big) \\
& \qquad \times (\|w\|_{\dot H^{2+\eta}} + 2\|\partial_t w\|_{\dot H^{1+\eta}})\,\mathrm{d}s \\
\le & \frac{1}{12}\big(\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) \\
& + C\|\partial_t v\|^2_{L^\infty([0,t],L^\infty)}\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) \\
& + C\|u\|^2_{L^\infty([0,t],L^\infty)}\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)}.
\end{aligned}
$$

Then, using the assumptions (4.2) and (4.3), we infer from the above inequality that, for $0 \le t \le \tau$,

$$
\begin{aligned}
|III_1^*(t)| \le & \frac{1}{12}\big(\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) \\
& + C\delta\big(\|\Delta v\|^2_{L^2([0,t],L^2)} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\nabla\partial_t v\|^2_{L^2([0,t],L^2)}\big).
\end{aligned}
\tag{4.35}
$$

The estimate of the term III_2^* is similar to the one of III_1^*. Applying the inequality (3.9) as well as Lemma 3.1 and using the smallness assumptions (4.2) and (4.3), we obtain, for $0 \le t \le \tau$,

$$
\begin{aligned}
|III_2^*(t)| \le & \int_0^t \|u\nabla\partial_t w\|_{\dot H^\eta}\|w\|_{\dot H^{2+\eta}}\,\mathrm{d}s \\
\le & C\int_0^t \big(\|u\|_{L^\infty}\|\partial_t w\|_{\dot H^{1+\eta}} + \|\partial_t w\|_{L^\infty}\|u\|_{\dot H^{1+\eta}}\big)\|w\|_{\dot H^{2+\eta}}\,\mathrm{d}s \\
\le & C\int_0^t \Big(\|u\|_{L^\infty}\|\partial_t w\|_{\dot H^{1+\eta}} + \|\partial_t w\|_{\dot H^{1+\eta}}\big(\|\nabla v\| + \|w\|_{\dot H^{1+\eta}}\big)\Big)\|w\|_{\dot H^{2+\eta}}\,\mathrm{d}s \\
\le & \frac{1}{12}\big(\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) + C\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\|u\|^2_{L^\infty([0,t],L^\infty)} + 2\|\nabla v\|^2_{L^\infty([0,t],L^\infty)} + 2\|w\|^2_{L^\infty([0,t],\dot H^{1+\eta})}\Big)\\
&\le \frac{1}{12}\big(\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})} + \|w\|^2_{L^2([0,t],\dot H^{2+\eta})}\big) + C\delta\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\,.
\end{aligned}
\tag{4.36}
$$

It remains to estimate the last term III_3^*, which is more involved to adress. We first recall here that, as explained in Remark 4.1, we are allowed to assume that $(w, \partial_t w)$ is regular and, in particular, that it belongs to the space $(H^{3+\eta}(\mathbb{R}^2))^2 \times (H^{2+\eta}(\mathbb{R}^2))^2$. This allows to give a sense to the expression III_3^* and to write the following equality:

$$
\Lambda^{1+\eta}(u\nabla\partial_t w) = [\Lambda^{1+\eta}, u]\nabla\partial_t w + u\Lambda^{1+\eta}(\nabla\partial_t w). \tag{4.37}
$$

Since Λ^s commutes with the derivatives and div $u = 0$, an integration by parts gives

$$
(u\Lambda^{1+\eta}(\nabla\partial_t w), \Lambda^{1+\eta}(\partial_t w)) = 0. \tag{4.38}
$$

From the properties (4.37) and (4.38) and the commutator inequality (4.22), we deduce

$$
\begin{aligned}
|III_3^*(t)| &\le C\int_0^t \|\nabla u\|_{\dot H^{1+\eta}}\|\nabla\partial_t w\|_{\dot H^{\eta}}\|\partial_t w\|_{\dot H^{1+\eta}}\,\mathrm{d}s\\
&\le C\int_0^t \big(\|\Delta v\| + \|w\|_{\dot H^{2+\eta}}\big)\|\partial_t w\|^2_{\dot H^{1+\eta}}\,\mathrm{d}s\\
&\le \frac{1}{12}\big(\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\Delta v\|^2_{L^2([0,t],L^2)}\big)\\
&\quad + C\|\partial_t w\|^2_{L^\infty([0,t],\dot H^{1+\eta})}\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}.
\end{aligned}
$$

Thus, thanks to the smallness assumptions (4.2) and (4.3), we have, for $0 \le t \le \tau$,

$$
\begin{aligned}
|III_3^*(t)| \le& \frac{1}{12}\big(\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\Delta v\|^2_{L^2([0,t],L^2)}\big)\\
&+ C\delta\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}.
\end{aligned}
\tag{4.39}
$$

Consequently, the estimates (4.35) to (4.39) imply, for $0 \le t \le \tau$,

$$
\begin{aligned}
|III^*(t)| \le& \left(\frac{1}{6} + C\delta\right)\|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\\
&+ \left(\frac{1}{12} + C\delta\right)\|\Delta v\|^2_{L^2([0,t],L^2)}\\
&+ \left(\frac{1}{4} + C\delta\right)\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + C\delta\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)}.
\end{aligned}
\tag{4.40}
$$

We then go back to the equality (4.24) and add (4.25), (4.29), (4.34), and (4.40). We get, for $0 \le t \le \tau$,

$$
\begin{aligned}
&E_w(t) + \frac{1}{2}\big(\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big)\\
&\le E_w(0) + C\|(I-\Pi_1)f\|^2_{L^2([0,t],\dot H^{1+\eta})} + C\delta\big(\|\nabla\partial_t v\|^2_{L^2([0,t],L^2)} + \|\Delta v\|^2_{L^2([0,t],L^2)}\big)\\
&\qquad + C\delta\big(\|w\|^2_{L^2([0,t],\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2([0,t],\dot H^{1+\eta})}\big),
\end{aligned}
$$

If we choose δ such that $C\delta \le 1/16$, we deduce (4.23) from the previous inequality, and achieve the estimate of the high frequencies of $(u, \partial_t u)$. □

4.2 Estimates in $H^1_\sigma \times L^2(\mathbb{R}^2)^2$

We now show that the $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$-norm of $(v, \partial_t v)$ remains bounded on the time interval $[0, \tau]$, provided the smallness assumption (4.2) is satisfied. Let E^* be the functional

$$E^*(t) = \frac{1}{2}\left(\|v(t) + \partial_t v(t)\|^2 + \|\partial_t v(t)\|^2\right) + \|\nabla v(t)\|^2 .$$

The estimates on the non-homogeneous part of $(v, \partial_t v)$ is stated in the following lemma.

Lemma 4.3 *Let $0 < \eta < 1$ and (u_0, u_1) and f satisfy the assumptions of Theorem 2.1. Assume that the smallness assumption (4.2) is satisfied. Then, there exists $\delta_0 > 0$ such that, if $\delta < \delta_0$, then, for all $t \in [0, \tau]$, the following energy estimate holds:*

$$\begin{aligned} E^*(t) &+ \frac{1}{4}\int_0^t (\|\nabla v(s)\|^2\,ds + \|\partial_t v(s)\|^2)\,ds \\ &\le E^*(0) + \frac{1}{12}\left(\|v\|^2_{L^\infty((0,t),L^2)} + \|\partial_t v\|^2_{L^\infty((0,t),L^2)}\right) \\ &\quad + C\|\Pi_1 f\|^2_{L^1((0,t),L^2)} + C\delta\left(\|w\|^2_{L^2((0,t),\dot H^{2+\eta})} + \|\partial_t w\|^2_{L^2((0,t),\dot H^{1+\eta})}\right), \end{aligned} \tag{4.41}$$

where $C > 0$.

Proof Taking the L^2-inner product of (4.4) with $v + 2\partial_t v$ and integrating in time over $[0, t]$, we obtain

$$E^*(t) + \int_0^t \|\nabla v(s)\|^2\,\mathrm{d}s + \int_0^t \|\partial_t v(s)\|^2\,\mathrm{d}s = E^*(0) + L_1 + L_2 + L_3 + L_4\,, \tag{4.42}$$

where

$$\begin{aligned} L_1 &= \int_0^t \big(u.\nabla v + \partial_t(u.\nabla v),\, (v + 2\partial_t v)\big)(s)\,\mathrm{d}s, \\ L_2 &= \int_0^t \big(u.\nabla w,\, (v + 2\partial_t v)\big)(s)\,\mathrm{d}s, \\ L_3 &= \int_0^t \big(\partial_t(u.\nabla w),\, (v + 2\partial_t v)\big)(s)\,\mathrm{d}s, \\ L_4 &= \int_0^t \big(\Pi_1(f),\, (v + 2\partial_t v)\big)(s)\,\mathrm{d}s, \end{aligned}$$

The estimate of the term L_4 is straightforward. Applying a Young inequality, we can write

$$|L_4(t)| \le \frac{1}{12}\left(\|v\|^2_{L^\infty((0,t),L^2)} + \|\partial_t v\|^2_{L^\infty((0,t),L^2)}\right) + C\|\Pi_1 f\|^2_{L^1((0,t),L^2)} \tag{4.43}$$

Since $(u.\nabla v, v)_{L^2} = 0 = \partial_t((u.\nabla v, v)_{L^2}) = 0$, the term L_1 reduces to

$$L_1 = \int_0^t \big((u.\nabla v, \partial_t v) + 2(\partial_t u \nabla v, \partial_t v)\big)\,\mathrm{d}s.$$

Using the above equality as well as a Young inequality, we obtain

$$\begin{aligned} |L_1(t)| \le \frac{1}{12}\|\nabla v\|^2_{L^2((0,t),L^2)} &+ C\|\partial_t v(t)\|^2_{L^2((0,t),L^2)}\big(\|u\|^2_{L^\infty((0,t),L^\infty)} \\ &+ \|\partial_t u\|^2_{L^\infty((0,t),L^\infty)}\big). \end{aligned} \tag{4.44}$$

Taking into account the smallness hypotheses (4.2) and (4.3), we then deduce

$$|L_1(t)| \leq \frac{1}{12}\|\nabla v\|^2_{L^2((0,t),L^2)} + C\delta\|\partial_t v(t)\|^2_{L^2((0,t),L^2)}. \tag{4.45}$$

Let us now estimate the term L_2. Noticing that $u.\nabla w = \nabla(u \otimes w)$ and applying Lemma 3.1 several times, we obtain

$$\begin{aligned} |L_2(t)| &\leq \int_0^t \Big(2\|u\|_{L^\infty}\|\nabla w\|\|\partial_t v\| + \|u \otimes w\|\|\nabla v\|\Big)\,\mathrm{d}s \\ &\leq \int_0^t \Big(2\|u\|_{L^\infty}\|\nabla w\|\|\partial_t v\| + \|u\|_{L^\infty}\|w\|\;\|\nabla v\|\Big)\,\mathrm{d}s \\ &\leq C\int_0^t \|u\|_{L^\infty}\|w\|_{\dot{H}^{1+\eta}}\big(\|\nabla v\| + \|\partial_t v\|\big)\,\mathrm{d}s. \end{aligned} \tag{4.46}$$

The estimate (4.46), together with the smallness hypotheses (4.2) and (4.3), imply, for all $t \in [0,\tau]$,

$$|L_2(t)| \leq \frac{1}{12}\big(\|\nabla v\|^2_{L^2((0,t),L^2)} + \|\partial_t v(t)\|^2_{L^2((0,t),L^2)}\big) + C\delta\|w\|^2_{L^2((0,t),\dot{H}^{2+\eta})}. \tag{4.47}$$

It remains to estimate L_3. Arguing like we did for L_2, we get

$$\begin{aligned} |L_3(t)| &\leq \int_0^t \big(\|\partial_t u \otimes w\| + \|u \otimes \partial_t w\|\big)\big(\|\nabla v\| + 2\|\nabla\partial_t v\|\big)\,ds \\ &\leq C\int_0^t \big(\|\partial_t u\|_{L^\infty}\|w\|_{\dot{H}^{1+\eta}} + \|u\|_{L^\infty}\|\partial_t w\|_{\dot{H}^{1+\eta}}\big)\big(\|\nabla v\| + \|\partial_t v\|\big)\,ds \end{aligned} \tag{4.48}$$

The inequality (4.48), together with the smallness assumptions (4.2) and (4.3), imply, for $0 \leq t \leq \tau$,

$$\begin{aligned} |L_3(t)| \leq &\frac{1}{12}\big(\|\nabla v\|^2_{L^2((0,t),L^2)} + \|\partial_t v(t)\|^2_{L^2((0,t),L^2)}\big) \\ &+ C\delta\big(\|w\|^2_{L^2((0,t),\dot{H}^{2+\eta})} + \|\partial_t w\|^2_{L^2((0,t),\dot{H}^{1+\eta})}\big). \end{aligned} \tag{4.49}$$

We next go back to the equality (4.42). Adding the inequalities (4.43), (4.45), (4.47), and (4.49), and choosing $\delta > 0$ so that $C\delta \leq 1/12$, we finally get the inequality (4.41). □

4.3 End of the Proof of Theorem 2.2

In this section, we sum up the estimates performed in Sects. 4.1 and 4.2 and deduce an estimate in the space $H^{2+\eta}_\sigma \times H^{1+\eta}_\sigma$. Then, a contradiction allows to conclude that Theorem 2.2 holds. By summing the inequalities (4.6), (4.23) and (4.41) and assuming that δ is small enough, we obtain, for all $t \in [0,\tau]$,

$$\begin{aligned} E_v(t) + E_w(t) + E^*(t) \leq &E_v(0) + E_w(0) + E^*(0) \\ &+ \frac{1}{12}\big(\|v\|^2_{L^\infty((0,t),L^2)} + \|\partial_t v\|^2_{L^\infty((0,t),L^2)}\big) \\ &+ c^*\Big(\|f\|^2_{L^1((0,\infty),L^2))} + \|f\|^2_{L^2((0,\infty),H^{1+\eta})}\Big), \end{aligned} \tag{4.50}$$

where $c^* > 0$. In particular, we have

$$\sup_{t\in[0,\tau]}\left(E_v(t)+E_w(t)+\frac{1}{2}E^*(t)\right)\leq E_v(0)+E_w(0)+E^*(0) + c^*\left(\|f\|^2_{L^1((0,\infty),L^2))}+\|f\|^2_{L^2((0,\infty),H^{1+\eta})}\right). \tag{4.51}$$

For later use, we set:

$$B=E_v(0)+E_w(0)+c^*\|f\|^2_{L^2((0,\infty),H^{1+\eta})}, \tag{4.52}$$

and

$$\begin{aligned} B^* =&E_v(0)+E_w(0)+c^*\|f\|^2_{L^2((0,\infty),H^{1+\eta})}+E^*(0)+c^*\|\Pi_1 f\|^2_{L^1((0,t),L^2)} \\ =&B+E^*(0)+c^*\|\Pi_1 f\|_{L^1((0,t),L^2)}. \end{aligned} \tag{4.53}$$

In the above estimates, we have several times used the smallness hypotheses (4.2) and (4.3), which involve L^∞-norms. We now intend to demonstrate that these L^∞-norms remain bounded on the time intervall $[0,\tau]$. Applying the Agmon inequality (4.16), we deduce from the estimate (4.51) that, for $0\leq t\leq\tau$,

$$\begin{aligned} \|u(t)\|^2_{L^\infty} &\leq 2(\|v(t)\|^2_{L^\infty}+\|w(t)\|^2_{L^\infty})\leq C\|v(t)\|_{L^2}\|\Delta v\|_{L^2}+C\|w(t)\|^2_{H^{2+\eta}} \\ &\leq CB^{1/2}(B^*)^{1/2}+CB. \end{aligned} \tag{4.54}$$

Likewise, we have

$$\begin{aligned} \|\partial_t u(t)\|^2_{L^\infty} &\leq 2(\|\partial_t v(t)\|^2_{L^\infty}+\|\partial_t w(t)\|^2_{L^\infty}) \\ &\leq C\|\partial_t v(t)\|_{L^2}\|\Delta\partial_t v\|_{L^2}+C\|\partial_t w(t)\|^2_{H^{1+\eta}} \\ &\leq C\|\partial_t v(t)\|_{L^2}\|\nabla\partial_t v\|_{L^2}+C\|\partial_t w(t)\|^2_{H^{1+\eta}} \\ &\leq CB^{1/2}(B^*)^{1/2}+CB. \end{aligned} \tag{4.55}$$

Besides, we get

$$\begin{aligned} \|\nabla u(t)\|^2_{L^\infty} &\leq 2(\|\nabla v(t)\|^2_{L^\infty}+\|\nabla w(t)\|^2_{L^\infty}) \\ &\leq C\big(\|\nabla v(t)\|^2+\|\Delta v(t)\|^2+\|w(t)\|^2_{\dot H^{2+\eta}}) \\ &\leq CB \end{aligned} \tag{4.56}$$

Finally, we recall that

$$\begin{aligned} \|\nabla u(t)\|^2+&\|\nabla\partial u(t)\|^2 \\ &\leq 2\big(\|\nabla v(t)\|^2+\|\nabla\partial_t v(t)\|^2+C\|w(t)\|^2_{\dot H^{2+\eta}}+C\|\nabla\partial_t w(t)\|^2_{\dot H^{1+\eta}}\big) \\ &\leq 2(1+C)B \end{aligned} \tag{4.57}$$

The estimates (4.54)–(4.57) imply, for $0\leq t\leq\tau$,

$$A(t)\leq C_1B+C_2B^{1/2}(B^*)^{1/2}, \tag{4.58}$$

where C_1 and C_2 are two positive constants.

We recall that T is the maximum time of existence of $(u,\partial_t u)$ and $\tau\leq T$. We will show that, if T is finite and the constant K_0 of Theorem 2.2 small enough, the smallness assumption on u_0, u_1 and f given by (2.4) leads to a contradiction. Indeed, if we choose K_0 small enough so that

$$C_1B+C_2B^{1/2}(B^*)^{1/2}\leq\delta, \tag{4.59}$$

Then, the property (4.58) implies, for $0 \le t \le \tau$,

$$A(t) \le \delta. \tag{4.60}$$

In particular, $A(t) < 2\delta$ for all $t \in [0, \tau]$ and, due to (4.3), we have $\tau = T$. Consequently, the estimate (4.51) implies that $\|u(t)\|_{\dot{H}^{2+\eta}} + \|\partial_t u(t)\|_{\dot{H}^{1+\eta}}$ is uniformly bounded on the time interval $[0, T]$, which contradicts the fact that T is finite. Therefore T is infinite and Theorem 2.2 is proved.

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