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par **Olivier Coulaud**

Dynamique de systèmes d'équations non-newtoniens

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Mme	Geneviève	RAUGEL	Université Paris Sud	Directrice de thèse
M.	Marius	PAICU	Université Bordeaux 1	Directeur de thèse
M.	Didier	BRESCH	Université de Savoie	Rapporteur
M.	Dragos	IFTIMIE	Université Lyon 1	Rapporteur
M.	Nicolas	BURQ	Université Paris Sud	Examinateur
M.	David	LANNES	École Normale Supérieure	Examinateur

Département de Mathématiques d'Orsay

Thèse préparée sous la direction de Geneviève Raugel et Marius Paicu
au Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8625), Bât. 425
Université Paris Sud
91405 Orsay Cedex

Résumé

Cette thèse a pour objet l'étude du comportement asymptotique des solutions des équations des fluides de grades 2 et 3. Dans le premier chapitre, on étudie les profils asymptotiques au premier ordre des solutions des équations des fluides de grade 2 sur $\mathbb{R}^3$. On démontre que les solutions des équations des fluides de grade 2 convergent vers des solutions particulières et explicites des équations de la chaleur, lorsque le temps tend vers l'infini. Ce résultat montre en particulier que les fluides de grade 2 se comportent asymptotiquement comme les fluides newtoniens régis par les équations de Navier-Stokes. Pour cette étude, on utilise les variables d'échelles (ou variables autosimilaires), et on effectue des estimations d'énergies dans divers espaces fonctionnels, en particulier dans des espaces de Sobolev à poids polynomiaux. La description des profils asymptotiques est obtenue sous des conditions de petitesse sur les données initiales de l'équation.

Le second chapitre de cette thèse traite des profils asymptotiques à l'ordre 1 des solutions des équations des fluides de grade 3 dans $\mathbb{R}^2$. À l'instar des résultats du premier chapitre, on obtient ici aussi la convergence des solutions de ces équations vers des solutions explicites des équations de la chaleur. Les outils utilisés pour cette étude sont semblables à ceux utilisés pour les fluides de grade 2 dans $\mathbb{R}^3$, à savoir les variables autosimilaires et des estimations d'énergies. Dans ce cas aussi, on conclut que les fluides de grade 3 se comportent asymptotiquement comme les fluides newtoniens.

Dans le dernier chapitre, on étudie l'existence d'un attracteur pour les équations des fluides de grade 3 en dimension 2 avec des conditions périodiques. On considère donc les solutions faibles de ces équations à données initiales dans l'espace de Sobolev H^1. Ces solutions faibles définissent un semi-groupe généralisé. Ensuite, on montre que les solutions à données initiales dans H^2 possèdent un attracteur global pour la topologie H^1. Pour ce travail, on utilise un schéma de Galerkin, des estimations a priori et une méthode de monotonie. Les principales difficultés que l'on rencontre sont liées au peu de régularité des données initiales et au fait que l'on ne sait par si les solutions des équations des fluides de grade 3 à données H^1 sont uniques.

Mots-clés : mécanique des fluides, dynamique des équations aux dérivées partielles, fluides de grade 2, fluides de grade 3, profils asymptotiques.

Abstract

This thesis is devoted to the study of the asymptotic behaviour of the solutions of the second and third grades fluids equations. In the first chapter, we study the asymptotic profiles to the first order of the solutions of the second grade fluids equations in $\mathbb{R}^3$. We show that these solutions behave asymptotically (when the time goes to infinity) like explicit solutions to the heat equations. This result shows in particular that the asymptotic behaviour of the fluids of grade 2 is the same as the one of the Newtonian fluids, modelized by the classical Navier-Stokes equations. For this study, we use scaled variables (also called self-similar variables), and we perform energy estimates in several functions spaces, including weighted Sobolev spaces. Notice that the first order asymptotic expansion that we obtain holds under smallness assumptions on the initial data.

In the second chapter of this thesis, we study the asymptotic profiles to the first order of the solutions of the third grade fluids equations in $\mathbb{R}^2$. As in the previous chapter, we establish the convergence of these solutions to explicit solutions to the heat equations. The methods that we use are very similar to the ones used in the case of the second grade fluids equations on $\mathbb{R}^3$, namely scaled variables and energy estimates. We also conclude that the fluids of grade 3 behave asymptotically in time like Newtonian fluids.

The last chapter is devoted to the study of the existence of an attractor for the third grade fluids equations in dimension 2 with periodic boundary conditions. We consider the weak solutions of these equations with initial data in H^1. These weak solutions define a generalized semiflow on H^1. Then, we show that the solutions with initial data in H^2 admit a global attractor for the H^1-topology. To this end, we use a Galerkin method, a priori estimates and a monotonicity method. The main difficulties come from the lack of regularity on the solutions and from the fact that these solutions are not known to be unique.

Key words : fluid mechanics, dynamics of partial differential equations, second grade fluids, third grade fluids, asymptotic profiles.

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En cas d'oubli, remplir le formulaire ci-dessous.

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- ☐ son soutien
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- ☐ autre : ....

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Chapitre 1

Introduction

L'objet de cette thèse est l'étude des comportements asymptotiques des solutions des équations des fluides de grades 2 et 3. Ces équations régissent une large classe de fluides dits non-newtoniens, dont les comportements ne pourraient pas être décrits par les équations classiques de Navier-Stokes, adaptées aux fluides newtoniens. L'intérêt de l'étude des fluides non-newtoniens est lié au fait que l'on trouve très souvent de tels fluides dans la nature ou dans l'industrie et que de nombreuses applications en découlent. Par exemple, certaines huiles utilisées dans l'industrie sont des fluides non-newtoniens, mais il existe également des exemples plus évidents, tels que du sable mouillé ou du fromage fondu. Dans ce premier chapitre, on introduit les équations de mouvement de ces fluides, en rappelant rapidement leur modélisation. Ensuite, nous parlerons plus en détail du comportement asymptotique des solutions de ces équations, sous deux aspects différents. Dans le cadre d'un fluide de grade 2 ou 3 remplissant tout l'espace $\mathbb{R}^2$ ou $\mathbb{R}^3$, on étudiera la convergence vers des solutions particulières appelées solutions autosimilaires. Dans le cadre d'un fluide de grade 3 sur un domaine périodique de $\mathbb{R}^2$, on s'intéressera à la convergence vers un domaine compact de l'espace dans lequel vivent les solutions de ces équations.

1 Équations du mouvement

Nous commençons par rappeler succinctement la modélisation des équations bien connues de Navier-Stokes, qui concernent les fluides newtoniens. Ces équations ont été introduites il y a bien longtemps et sont l'objet d'une littérature mathématique très fournie.

1.1 Fluides newtoniens et équations de Navier-Stokes

Soit Ω un domaine fixe de $\mathbb{R}^2$ ou $\mathbb{R}^3$, rempli par un fluide incompressible évoluant au cours du temps t dans ce domaine. Afin de modéliser ce fluide, on définit les trajectoires de chaque particule de fluide par la fonction

$$\begin{array}{rccc} \psi: & \mathbb{R}^+ \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^d, \\ & (t,x) & \longmapsto & \psi\left(t,x\right), \end{array}$$

où $\psi\left(t,x\right)$ représente la particule de fluide au temps t, qui était à la position x au temps $t = 0$. On suppose que ψ est continûment différentiable en t et x, et on définit le champ de vitesse $u(t,x)$ qui représente la vitesse de la particule $\psi(t,x)$. Formellement, ψ satisfait l'équation différentielle

$$\begin{aligned} &\partial_t \psi\left(t,x\right) = u\left(t, \psi\left(t,x\right)\right), \\ &\psi\left(0,x\right) = x. \end{aligned}$$

Si l'on suppose que u est suffisamment régulier, le théorème de Cauchy-Lipschitz permet par cette égalité de reconstituer ψ en supposant simplement u connu. Ainsi, on peut modéliser de façon équivalente le mouvement du fluide par la position ψ de ses particules ou par son champ de vitesses u. Dans cette thèse, nous privilégions la modélisation par le champ de vecteurs vitesses (représentation eulérienne). Étant donné que le fluide que l'on considère est incompressible, on a, pour tout $t \geq 0$,

$$\int_\Omega 1 dx = \int_{\psi(t,\Omega)} 1 dx.$$

En effectuant formellement le changement de variable $x = \psi\left(t,y\right)$, on obtient

$$\int_\Omega 1 dx = \int_\Omega \left|\det\left(\mathrm{Jac}\left(\psi\left(t,y\right)\right)\right)\right| dy.$$

De plus, cette propriété se vérifie quel que soit le domaine Ω. En prenant des domaines de plus en plus petits, on peut donc conclure que

$$\left|\det\left(\mathrm{Jac}\left(\psi\left(t,y\right)\right)\right)\right| = 1,$$

et par continuité en $t = 0$, on en déduit que $\det\left(\mathrm{Jac}\left(\psi\left(t,y\right)\right)\right) = 1$. En dérivant cette égalité en temps, on a donc

$$\partial_t\left(\det\left(\mathrm{Jac}\left(\psi\left(t,y\right)\right)\right)\right) = 0.$$

Or, un calcul simple montre que $\partial_t (\text{Jac}(\psi(t,x))) = \nabla u(t,x)\text{Jac}(\psi(t,x))$, où ∇u est la matrice définie par $(\nabla u)_{i,j} = \partial_j u_i$. Par application du théorème de Liouville, on obtient

$$0 = \partial_t (\det(\text{Jac}(\psi(t,y)))) = \det(\text{Jac}(\psi(t,x)))\,\text{Tr}(\nabla u) = \text{div}\, u.$$

Le caractère incompressible du fluide se traduit donc par l'égalité

$$\text{div}\, u = 0. \tag{1.1}$$

On note $\rho = \rho(t,x)$ la densité du fluide que l'on considère. L'équation de conservation de la masse, ou équation de continuité, nous donne

$$\partial_t \rho + \text{div}\ (\rho u) = 0. \tag{1.2}$$

Le fluide que l'on considère est supposé de densité uniforme en espace. Comme le fluide est également supposé incompressible, la propriété (1.1) et l'équation (1.2) impliquent que $\partial_t \rho = 0$, et la densité est donc constante à la fois en temps et en espace. On note, $v(t,\psi(t,x))$ l'accélération de la particule $\psi(t,x)$ au temps t. En particulier on a

$$v(t,\psi(t,x)) = \partial_t (u(t,\psi(t,x))) = (\partial_t u + u.\nabla u)(t,\psi(t,x)),$$

où $(u.\nabla u)_i = \displaystyle\sum_{j=1}^{d} u_j \partial_j u_i$.

L'équation de la conservation de quantité de mouvement (principe fondamental de la dynamique) implique

$$\rho \partial_t u(t,\psi(t,x)) = \rho(\partial_t u + u.\nabla u)(t,\psi(t,x)) = F + \text{div}\, \sigma, \tag{1.3}$$

où $\sigma \in \mathbb{R}^d \times \mathbb{R}^d$ est le tenseur des contraintes et F la somme des forces extérieures qui agissent sur le fluide. Le fluide que l'on modélise dans cette section est newtonien, ce qui veut dire que le tenseur des contraintes que l'on considère est une fonction affine du gradient du champ de vitesses u, on suppose donc

$$\sigma = -PId + \mu A,$$

où P est la pression du fluide, μ est la viscosité dynamique du fluide et A est le tenseur des déformations de u, donné par $A^{i,j} = \partial_j u_i + \partial_i u_j$. En revenant à l'équation (1.3) que l'on divise par la constante ρ, on obtient le système bien connu des équations de Navier-Stokes

$$\partial_t u - \nu \Delta u + u.\nabla u + \nabla p = f, \tag{1.4}$$

où $\nu = \dfrac{\mu}{\rho}$ est la viscosité cinématique, $p = \dfrac{P}{\rho}$ et $f = \dfrac{F}{\rho}$.

Le système d'équations (1.4) sert à modéliser des fluides tels que l'eau, l'air ou encore de nombreux gaz. Il existe énormément de travaux traitant de ces équations, dont les premiers résultats d'existence remontent aux années 1930 (voir par exemple [51] ou [32]).

1.2 Équations des fluides de grade 2

Introduites bien après les équations de Navier-Stokes, les équations des fluides de grade 2 modélisent une classe de fluides plus générale que celle des fluides newtoniens. La modélisation diffère de celle des équations de Navier-Stokes par le choix du tenseur des contraintes, qui n'est plus linéaire par rapport au gradient de la vitesse. Nous ne donnons que peu de détails sur la modélisation de ces fluides, mais on peut trouver plus de précisions et de justifications physiques dans l'article de J. Dunn et R. Fosdick [24] ou encore dans [59] et [62]. Les fluides de grade 2 appartiennent à une classe particulière de fluides non-newtoniens, à savoir les fluides différentiels de Rivlin-Ericksen, décrits dans [59]. Il convient de préciser que tous les fluides non-newtoniens ne sont pas des fluides différentiels, et que par conséquent on se restreint ici à une certaine classe de fluides non-newtoniens. Selon ce modèle, on définit les fluides de grade n pour lesquels le tenseur des contraintes est de la forme

$$\sigma = -PId + Q\left(A_1, A_2, ...A_n\right),$$

où Q est un polynôme de degré n et A_k est le $k^{\text{è}}$ tenseur de Rivlin-Ericksen, donné par la relation de récurrence

$$\begin{aligned}
&A_1 = \nabla u + (\nabla u)^t\,,\\
&A_k = \partial_t A_{k-1} + u.\nabla A_{k-1} + (\nabla u)^t A_{k-1} + A_{k-1}\nabla u.
\end{aligned}$$

Pour les fluides de grade 2, le tenseur des contraintes σ de l'équation (1.3) s'écrit sous la forme

$$\sigma = -PId + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

où μ est la viscosité dynamique, α_1 et α_2 sont deux réels et A_1 et A_2 sont les deux premiers tenseurs de Rivlin-Ericksen. Dans cette thèse, on considérera les équations des fluides de grade 2 telles qu'elles sont posées par J. Dunn et L. Fosdick. En particulier, des considérations physiques venant de la thermodynamique permettent de supposer

$$\alpha_1 + \alpha_2 = 0 \quad \text{et} \quad \mu \geq 0.$$

En revenant à l'équation (1.3) et en posant $\alpha = \alpha_1$, on obtient le système d'équations des fluides de grade 2

$$\partial_t (u - \alpha \Delta u) - \nu \Delta u + \text{rot } (u - \alpha \Delta u) \wedge u + \nabla p = f, \tag{1.5}$$

où $\wedge$ est le produit vectoriel classique de $\mathbb{R}^3$.

En dimension 2, on fait la convention que u est un vecteur de $\mathbb{R}^3$ dont la troisième composante est nulle, $u = (u_1, u_2, 0)$. Ainsi, le rotationnel de u s'écrit

$$\text{rot } u = \nabla \wedge u = (0, 0, \partial_1 u_2 - \partial_2 u_1),$$

ce qui permet de définir le système (1.5) en dimension 2.

1.3 Équations des fluides de grade 3

Une autre classe de fluides étudiée dans le cadre de cette thèse est la classe des fluides de grade 3. Ces fluides font eux aussi partie de la classe des fluides différentiels. Dans les résultats sur les fluides de grade 3 présentés dans cette thèse, on prendra en compte le modèle introduit en 1980 par R. Fosdick et K. Rajagopal dans [31]. Dans cet article, une étude thermodynamique amène à considérer le tenseur

$$\sigma = -PId + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta |A_1|^2 A_1,$$

où μ est la viscosité dynamique, α_1, α_2 et β sont des nombres réels et

$$|A_1| = \left(\sum_{i,j=1}^{d} \left(A_1^{i,j}\right)^2 \right)^{1/2}.$$

De plus, des considérations physiques permettent de justifier le fait que

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad \beta \geq 0 \quad \text{et} \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\nu\beta}. \tag{1.6}$$

En revenant à l'équation (1.3), on obtient finalement l'équation

$$\begin{aligned} \partial_t (u - \alpha_1 \Delta u) - \nu \Delta u + \text{rot } (u - \alpha_1 \Delta u) \wedge u \\ - (\alpha_1 + \alpha_2) \left(A \Delta u + 2\text{div } \left((\nabla u)^t \nabla u \right) \right) - \beta \text{div } \left(|A|^2 A \right) + \nabla p = f, \end{aligned} \tag{1.7}$$

où l'on a la convention $u = (u_1, u_2, 0)$ en dimension 2.

2 Comportements asymptotiques

Les résultats obtenus dans cette thèse décrivent les comportements asymptotiques en temps des solutions des équations des fluides de grades 2 et 3. On s'intéressera principalement à deux aspects différents de l'étude de comportements en grand temps, à savoir l'étude des profils asymptotiques et l'étude de l'existence d'un attracteur pour les systèmes des fluides de grades 2 et 3.

2.1 Profils asymptotiques

Une partie significative des résultats exposés dans cette thèse traite des profils asymptotiques des équations des fluides de grades 2 et 3. On considère un fluide de grade 2 ou 3 remplissant tout l'espace $\mathbb{R}^d$ et auquel aucune force extérieure n'est appliquée, ce qui revient à prendre $f = 0$ dans les systèmes (1.5) et (1.7). Dans les deux cas et comme on s'y attend, lorsqu'on laisse le temps s'écouler à l'infini, les fluides reviennent au repos et les solutions de ces équations tendent vers 0. Étudier les profils asymptotiques de ces équations, c'est en quelque sorte étudier la façon dont les solutions de ces dernières tendent vers 0. Le but recherché dans cette démarche est de montrer que le système d'équations que l'on considère se simplifie lorsque le temps devient grand, et est dominé par un système linéaire dont les solutions sont bien connues. L'intérêt de ce type de travaux réside dans le fait que, lorsque le temps est suffisamment grand, il devient pertinent d'approcher les solutions du système de départ par celles d'un système linéaire. Considérons une solution u d'un système d'équations aux dérivées partielles dont on ne connait pas explicitement la solution. Au premier ordre, l'idée générale est de décomposer u sous la forme

$$u(t, x) = \eta(t)G(t, x) + R(t, x), \tag{2.1}$$

où G est une fonction explicite, η est une fonction réelle dépendant du temps et de u et R est un reste qui tend vers 0 plus rapidement que ηG lorsque le temps tend vers l'infini. Dans la décomposition (2.1), l'intérêt semble limité par le fait que η dépende de u et donc du système de départ. En vérité, dans les cas dont on parlera plus tard, on montrera que η est solution d'une équation différentielle ordinaire dont la solution est explicite et dépend seulement des données initiales de l'équation. Les travaux présentés dans cette thèse portent sur les profils asymptotiques au premier ordre, c'est à dire que l'on obtient la décomposition (2.1). Cependant, on peut aussi s'intéresser aux profils asymptotiques à l'ordre supérieur, c'est-à-dire, pour $n \in \mathbb{N}$, décomposer la solution sous la forme

$$u(t, x) = \sum_{k=1}^{n} \eta_k(t)G_k(t, x) + R_n(t, x), \tag{2.2}$$

où pour tout k, la fonction G_k est explicite, η_k satisfait une équation différentielle ordinaire que l'on sait résoudre et R_n tend vers 0 plus rapidement que $\sum_{k=1}^{n} \eta_k(t) G_k(t,x)$. Pour tout k, le produit $\eta_k G_k$ de la décomposition (2.2) est appelé profil asymptotique de u à l'ordre k.

Afin d'illustrer l'idée générale des profils asymptotiques des solutions des fluides de grades 2 et 3, revenons au cas plus simple des fluides newtoniens en dimension 2. Ceci a été étudié par T. Gallay et E. Wayne aux ordres 1 et 2 dans [36] et [39]. Le cas de la dimension 3 est traité dans [37]. On considère donc un fluide newtonien remplissant tout l'espace $\mathbb{R}^2$ et u un champ de vecteurs à divergence nulle de $\mathbb{R}^2$, solution de l'équation (1.4). Dans les chapitres 3 et 4, on sera amené à considérer non pas directement les équations de mouvement (1.5) et (1.7) mais les équations satisfaites par le tourbillon $w = \text{rot } u$. En dimension 2, on utilise la convention

$$\text{rot } u = \partial_1 u_2 - \partial_2 u_1.$$

En prenant le rotationnel du système d'équations de Navier-Stokes (1.4), et en remarquant que

$$\text{rot } (u.\nabla u) = u.\nabla w,$$

les équations du mouvement des fluides newtoniens en dimension 2 deviennent

$$\partial_t w - \Delta w + u.\nabla w = 0, \tag{2.3}$$

où l'on a supposé $\nu = 1$.

Cette nouvelle équation, où w est l'inconnue, est bien autonome. En effet, le champ de vecteurs à divergence nulle u est reconstitué à partir de w par la loi de Biot-Savart, qui est un outil assez classique en mécanique des fluides. En particulier, passer en tourbillon dans les équations de Navier-Stokes permet de supposer que u n'est pas dans l'espace de Lebesgue L^2. Étant donné w une fonction de $\mathbb{R}^2$, on définit u par la formule

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} w(y) dy,$$

où $(x_1, x_2)^{\perp} = (-x_2, x_1)$.

Pour peu que w soit suffisamment régulière (par exemple dans un espace L^p avec $p < 2$),

u est bien définie et l'on a bien rot $u = w$ et div $u = 0$.

L'existence de solutions à l'équation (2.3) a été très largement étudiée par le passé et il existe notamment une classe de solutions particulières dites autosimilaires, c'est-à-dire de la forme

$$(t, x) \mapsto \frac{1}{1+t} F\left(\frac{x}{\sqrt{1+t}}\right),$$

où F est une fonction réelle sur $\mathbb{R}^2$ et T une constante positive.

Afin d'obtenir une décomposition de w sous la forme (2.1), on procède à un changement de variable qui revient à écrire les solutions de (2.3) dans les coordonnées de la fonction F. On pose $X = \dfrac{x}{\sqrt{1+t}}$ et $\tau = \ln(1+t)$ et on définit W comme suit :

$$\begin{aligned} w(t,x) &= \frac{1}{1+t} W\left(\ln(1+t), \frac{x}{\sqrt{1+t}}\right), \\ W(\tau, X) &= e^{\tau} w\left(e^{\tau} - 1, e^{\tau/2} X\right). \end{aligned} \tag{2.4}$$

Dans ce nouveau système de variables, appelées variables d'échelles ou variables autosimilaires, W satisfait l'équation

$$\partial_\tau W - \mathcal{L}(W) + U.\nabla W = 0, \tag{2.5}$$

où $\mathcal{L}(W) = W + \Delta W + \frac{X}{2}.\nabla W$.

L'idée est maintenant de décomposer W sur le spectre de $\mathcal{L}$. Dans des espaces fonctionnels bien choisis (en l'occurence des espaces de Lebesgue à poids), le spectre de cet opérateur est entièrement connu, de même que les vecteurs propres associés aux valeurs propres de ce dernier. Notamment, 0 est la plus grande valeur propre de $\mathcal{L}$, de multiplicité 1. Le vecteur qui lui est associé est appelé tourbillon d'Oseen et est donné par

$$G(X) = \frac{1}{4\pi} e^{-\frac{|X|^2}{4}}.$$

Ainsi, en supposant que W appartient à un espace bien choisi et en projetant W sur l'espace propre associé à la valeur propre 0, on peut montrer que W satisfait la décomposition

$$W(\tau, X) = \eta(\tau) G(X) + R(\tau, X),$$

où $\eta(\tau) = \int_{\mathbb{R}^2} W(\tau, x)dx$ et $R(\tau, X)$ tend vers 0 exponentiellement lorsque τ tend vers l'infini. En intégrant en espace l'équation (2.3), on constate que

$$\partial_\tau \eta = 0.$$

De ce fait, on a l'égalité $\eta(\tau) = \eta = \int_{\mathbb{R}^2} W(0, x)dx$. Étant donné que le changement de variable (2.4) préserve la masse totale, on a

$$\eta = \int_{\mathbb{R}^2} w(0, x)dx.$$

En revenant dans les variables de départ, on obtient une décomposition de la forme (2.1), dans notre cas

$$w(t, x) = \eta \frac{1}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) + r(t, x),$$

où $r(t, x)$ tend vers 0 de façon polynomiale lorsque t tend vers l'infini, plus rapidement que $\frac{1}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right)$. Il existe également des résultats sur les profils d'ordre supérieur pour ces équations, en décomposant W sur les espaces propres de $\mathcal{L}$ associés aux valeurs propres suivantes (voir [36]). On verra plus tard que l'étude des profils asymptotiques à l'ordre supérieur requiert des restrictions sur les espaces fonctionnels dans lesquels on considère les solutions.

2.2 Attracteurs

Un autre aspect du comportement asymptotique de solutions d'équations aux dérivées partielles est l'étude de l'existence d'un attracteur. Dans cette thèse, ce sujet est abordé au chapitre 4 pour les équations des fluides de grade 3 sur un domaine borné périodique de $\mathbb{R}^2$, où la force f que l'on applique au système (1.7) est supposée constante au cours du temps. On montre dans ce cas l'existence d'un attracteur pour une topologie plus faible que celle dans laquelle les solutions sont définies. Pour cette étude, on associe les solutions des équations des fluides de grade 3 à un système dynamique $S(t)$ sur un espace fonctionnel X de dimension infinie. L'étude asymptotique de ces solutions se fait alors par l'intermédiaire de l'étude des propriétés du système dynamique. Le fait qu'un système d'équations différentielles admette un attracteur global se traduit par le fait que les solutions de ce système sont attirées vers un ensemble compact de X, invariant par rapport au système dynamique $S(t)$. D'un point de vue plus concret, cela veut dire que lorsqu'on laisse le temps s'écouler avec une force constante agissant sur le système, le comportement du fluide se stabilise autour de cet ensemble invariant de solutions.

Considérons (X,d) un espace métrique donné et $S(t)$ un système dynamique sur X. Dans le cas d'un système dynamique associé aux solutions d'une équation d'évolution, l'espace X est l'espace dans lequel sont les données initiales de ces solutions. Ainsi, étant donné $x_0 \in X$, $S(t)x_0$ est la solution de donnée initiale x_0, prise à l'instant t. Pour un espace métrique donné (X,d), on définit la semi-distance sur les sous-espaces de X donnée par

$$\delta_X(A,B) = \sup_{b\in B}\inf_{a\in A} d\left(a,b\right).$$

Cette semi-distance, qui n'est pas symétrique, va jouer un rôle important par la suite. Elle mesure en quelque sorte "à quel point l'ensemble A est inclus dans l'ensemble B". En particulier, si $A \subset B$, alors $\delta_X(A,B) = 0$. Un attracteur global sur X pour le système dynamique $S(t)$ est un ensemble compact $\mathcal{A}$ de X, invariant par $S(t)$ et tel que, pour tout borné B de X et tout $\varepsilon \geq 0$, il existe un temps $t_\varepsilon = t_\varepsilon(B) \geq 0$ tel que, pour tout $t \geq t_\varepsilon$, on a

$$\delta_X\left(S(t)B, \mathcal{A}\right) \leq \varepsilon. \tag{2.6}$$

Généralement, et comme ce sera le cas dans les résultats présentés dans cette thèse, le système dynamique associé aux solutions d'une équation aux dérivées partielles agit sur un espace métrique de dimension infinie, typiquement un espace de fonctions. Ce type d'étude a été initié notamment par les travaux de J. Hale J. Lasalle et M. Slemrod (voir [44] et [43]). L'intérêt de montrer l'existence d'un attracteur pour les solutions d'un système d'équations est lié à la nature de cet attracteur. La dynamique d'un système au sein même de l'attracteur est une question importante de ce type d'étude. Existe-t-il des points d'équilibre ? Y en a t-il plusieurs ? Existe-t-il des solutions périodiques ? L'attracteur est-il plus régulier que l'espace dans lequel les solutions sont définies ? Chacune des réponses à ces questions donne des informations sur le comportement asymptotique des solutions du système. Par exemple, pour les équations des fluides de grade 2 sur un domaine borné ou périodique de $\mathbb{R}^2$, l'existence d'un attracteur pour des données dans les espaces de Sobolev $W^{3,p}$, avec $1 < p < +\infty$ est connue pour une force f indépendante du temps dans $W^{1,p}$ (voir [53], [57] ou [56]). De plus il a également été montré par M. Paicu et G. Raugel que l'attracteur est de régularité $W^{3,p+m}$, où m est une constante strictement positive, si l'on suppose que f est constante en temps dans l'espace $W^{1,p+m}$ (voir [56]).

Chapitre 2

Historique des résultats et contributions de la thèse

Dans le cadre de cette thèse, on s'intéresse aux comportements asymptotiques des solutions de deux systèmes d'équations de fluides non-newtoniens, à savoir les équations des fluides de grades 2 et 3. Dans ce chapitre, on donne un aperçu des résultats existants pour ces différentes classes de fluides, et on détaille les contributions de cette thèse. Tout d'abord, nous rappelons ce qui est connu pour les équations classiques de Navier-Stokes, ce qui nous permettra ensuite de comparer avec les résultats obtenus pour les fluides de grades 2 et 3.

1 Comportement asymptotique des fluides newtoniens

Le cas des fluides newtoniens, régis par les équations de Navier-Stokes, a été très largement étudié (voir par exemple [13], [27], [29], [36], [37], [38], [39], [14], [42], [49]). Dans cette section, on considère le problème de Cauchy des équations de Navier-Stokes en dimension d, où $d \in \{2, 3\}$, donné par

$$\begin{aligned} &\partial_t u - \nu \Delta u + u.\nabla u + \nabla p = f, \\ &\text{div } u = 0, \\ &u_{|t=0} = u_0, \end{aligned} \tag{1.1}$$

où $\nu > 0$ est la viscosité cinématique introduite dans le chapitre 1, p est la pression et f la force extérieure agissant sur le fluide.

Les résultats d'existence les plus connus pour ce système sont ceux de J. Leray [51] et H. Fujita et T. Kato [32]. Sur l'espace entier $\mathbb{R}^d$ (ou sur un domaine borné de $\mathbb{R}^d$ avec conditions de Dirichlet), J. Leray a démontré l'existence de solutions faibles globales en temps à valeurs dans $L^2(\mathbb{R}^d)$, dites solutions de Leray, pour une force f appartenant à l'espace $L^2\left(\mathbb{R}^+, H^{-1}(\mathbb{R}^d)\right)$. L'unicité de ces solutions est connue en dimension 2 mais reste un problème ouvert en dimension 3. Des éléments de réponse sont apportés en dimension 3 par le théorème de Fujita-Kato, qui montre l'existence de solutions fortes dans les espaces de Sobolev homogènes. Pour $s \in \mathbb{R}$, on définit la semi-norme

$$\|u\|_{\dot{H}^s} = \left\|\bar{\mathcal{F}}\left(|\xi|^s \widehat{u}\right)\right\|_{L^2},$$

où $\widehat{u}$ est la transformée de Fourier de u et $\bar{\mathcal{F}}$ désigne la transformée de Fourier inverse. On définit les espaces fonctionnels

$$\mathcal{V} = \left\{u \in C_0^\infty(\mathbb{R}^d) : \operatorname{div} u = 0\right\}, \text{ et } \dot{V}^s = \overline{\mathcal{V}}^{\|\cdot\|_{\dot{H}^s}}.$$

Dans [32], il est montré que si la donnée u_0 appartient à l'espace $\dot{V}^{\frac{1}{2}}(\mathbb{R}^3)$ et la force f à l'espace $L^2\left(\mathbb{R}^+, \dot{V}^{-\frac{1}{2}}(\mathbb{R}^3)\right)$, il existe alors une constante positive T et une solution forte u au système (1.1), telles que

$$u \in C^0\left([0,T], \dot{V}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T], \dot{V}^{\frac{3}{2}}(\mathbb{R}^3)\right).$$

Cette solution est de plus unique et globale en temps si la donnée initiale et la force sont suffisamment petites dans $\dot{V}^{\frac{1}{2}}(\mathbb{R}^3)$ et $L^2\left(\mathbb{R}^+, \dot{V}^{-\frac{1}{2}}(\mathbb{R}^3)\right)$ respectivement. L'existence de solutions globales dans $\dot{V}^{\frac{1}{2}}(\mathbb{R}^3)$ pour des données grandes est aujourd'hui encore un problème ouvert.

1.1 Profils asymptotiques

Il existe de nombreux travaux traitant du comportement asymptotique des solutions des équations de Navier-Stokes, et nous considérons ici la description des profils asymptotiques des solutions de ces équations. Nous nous intéressons dans un premier temps au cas de la dimension 2.

Profils asymptotiques en dimension 2

On considère un fluide newtonien remplissant l'espace $\mathbb{R}^2$ tout entier et auquel on n'applique pas de force extérieure. Pour étudier les profils asymptotiques des équations de Navier-Stokes, on s'intéresse aux équations du tourbillon $w = \text{rot}\, u = \partial_1 u_2 - \partial_2 u_1$, données par

$$\begin{aligned} &\partial_t w - \Delta w + u.\nabla w = 0, \\ &\text{div}\, u = 0, \\ &w_{|t=0} = w_0, \end{aligned} \tag{1.2}$$

où l'on a supposé $\nu = 1$ et le champ de vecteurs u est reconstruit à partir de w par la loi de Biot-Savart bidimensionnelle, donnée par

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} w(y) dy. \tag{1.3}$$

Cette étude se fait pour des solutions du système (1.2) à valeurs dans des espaces de Lebesgue à poids. On verra plus tard que ces espaces apparaissent naturellement lorsque l'on considère les variables d'échelles dont on a parlé dans l'introduction de cette thèse. Pour $m \in \mathbb{N}$, on définit

$$L^2(m) = \left\{ u \in L^2(\mathbb{R}^2) : \left(1 + |x|^2\right)^{m/2} u \in L^2(\mathbb{R}^2) \right\},$$

équipé de la norme

$$\|u\|_{L^2(m)} = \left\| \left(1 + |x|^2\right)^{m/2} u \right\|_{L^2}. \tag{1.4}$$

Il a été montré en 2002 par T. Gallay et E. Wayne dans [36] que si les données initiales sont suffisamment petites dans $L^2(m)$, les solutions de (1.2) convergent, à une constante dépendant des données initiales près, vers une solution autosimilaire de l'équation de la chaleur

$$\partial_t u - \Delta u = 0. \tag{1.5}$$

Cette solution particulière est définie par

$$(t, x) \longmapsto \frac{1}{1+t} G\left(\frac{x}{\sqrt{1+t}} \right), \tag{1.6}$$

où G est le tourbillon d'Oseen, donné par

$$G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}. \tag{1.7}$$

Ce résultat s'obtient en effectuant le changement de variables $X = \dfrac{x}{\sqrt{1+t}}$ et $\tau = \ln(1+t)$ et en définissant W et U comme suit :

$$\begin{aligned} w(t,x) &= \frac{1}{1+t} W\left(\ln(1+t), \frac{x}{\sqrt{1+t}}\right), \\ u(t,x) &= \frac{1}{\sqrt{1+t}} U\left(\ln(1+t), \frac{x}{\sqrt{1+t}}\right). \end{aligned} \tag{1.8}$$

Ces nouvelles variables s'appellent variables d'échelles ou variables autosimilaires, qui ont été initialement introduites pour étudier les comportements asymptotiques de solutions d'équations paraboliques, et en particulier leur convergence vers des solutions autosimilaires (voir [25], [26], [33] ou [48]). Nous verrons que ces variables permettent aussi de traiter le comportement asymptotique de solutions d'équations qui ne sont pas paraboliques. En revenant à l'équation (1.2), on peut vérifier que W satisfait

$$\partial_\tau W - \mathcal{L}W + U.\nabla W = 0, \tag{1.9}$$

où $\mathcal{L}W = W + \Delta W + \frac{X}{2}.\nabla W$.

Une étude du spectre de $\mathcal{L}$ sur $\mathbb{L}^2(m)$ montre que celui-ci est la réunion du spectre discret

$$\sigma_d = \left\{-\tfrac{k}{2} : k \in \{0, ..., m-2\}\right\},$$

et du spectre continu

$$\sigma_c = \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\tfrac{m-1}{2}\right\}.$$

En particulier, si $m \geq 2$, la valeur propre 0 est simple et isolée, et G en est un vecteur propre. De plus, il est facile de vérifier que G est solution de (1.9). On décompose W comme suit :

$$W = \eta G + R,$$

où η est une constante dépendant de W. Une étude plus détaillée du supplémentaire de l'espace propre associé à la valeur propre 0 montre que $\eta = \displaystyle\int_{\mathbb{R}^2} W(\tau, X)dX$. En intégrant en espace l'égalité (1.9), on constate que η est constant au cours du temps et donc $\eta = \displaystyle\int_{\mathbb{R}^2} W(0, X)dX$. De plus, cette quantité est conservée par le changement de variable (1.6), et on a donc $\eta = \displaystyle\int_{\mathbb{R}^2} w_0(x)dx$.

En utilisant des arguments provenant de l'étude des systèmes dynamiques, T. Gallay et E. Wayne ont ensuite montré que R tend exponentiellement vers 0 lorsque le temps τ tend vers l'infini. Pour $m \geq 2$, les auteurs ont montré, en reprenant notamment les travaux de X. Chen, J. Hale et B. Tan dans [16], l'existence d'une sous-variété W_c localement invariante par le flot associé à (1.9) sur $L^2(m)$. Leurs résultats montrent de plus que le comportement des solutions de (1.9) sur cette variété est déterminé par leurs projections sur les espaces propres associés aux valeurs propres isolées de $\mathcal{L}$. Ensuite, toujours en s'appuyant sur des résultats de [16], ils ont montré que, si les données initiales sont suffisamment petites dans $L^2(m)$, les solutions de (1.9) tendent vers cette sous-variété lorsque le temps tend vers l'infini. En particulier, si $m = 2$, ils ont montré qu'il existe une constante positive r_0 telle que les solutions de (1.9) tendent vers

$$W_c^{loc} = \left\{ AG : A \in \mathbb{R}, |A| \, \|G\|_{L^2(2)} < r_0 \right\},$$

qui est en fait la restriction de W_c à un voisinage de l'origine. Ce résultat est énoncé dans le théorème suivant, que l'on retrouve dans [36].

Théorème 1.1 *Soit $0 < \mu < \frac{1}{2}$ une constante fixée. Il existe $r > 0$ telle que, pour toute donnée initiale $W_0 \in L^2(2)$ avec $\|W_0\|_{L^2(2)} \leq r$, il existe une unique solution $W \in C^0(\mathbb{R}^+, L^2(2))$ de (1.9) avec $W(0) = W_0$ et une constante positive C telles que, pour tout $\tau \geq 0$,*

$$\|W(\tau) - \eta G\|_{L^2(2)} \leq Ce^{-\mu\tau}, \tag{1.10}$$

où $\eta = \int_{\mathbb{R}^2} W_0(X)dX = \int_{\mathbb{R}^2} w_0(x)dx$.

Dans les variables de départ, on obtient le théorème suivant, qui est un corollaire du théorème 1.1.

Théorème 1.2 *Soit $0 < \mu < \frac{1}{2}$ une constante fixée. Il existe $r > 0$ telle que, pour toute donnée initiale $w_0 \in L^2(2)$ avec $\|w_0\|_{L^2(2)} \leq r$, il existe une unique solution $w \in C^0(\mathbb{R}^+, L^2(2))$ de (1.2). De plus, pour tout $1 \leq p \leq 2$, il existe une constante positive C_p telle que, pour tout $t \geq 0$,*

$$\left\| w(t) - \frac{\eta}{1+t} G\left(\frac{.}{\sqrt{1+t}}\right) \right\|_{L^p} \leq \frac{C_p}{(1+t)^{1+\mu-\frac{1}{p}}}, \tag{1.11}$$

où $\eta = \int_{\mathbb{R}^2} w_0(x)dx$. Si u est le champ de vitesses obtenu à partir de w par la loi de Biot-Savart, alors, pour tout $1 < q < \infty$, il existe une constante positive C_q telle que, *pour tout $t \geq 0$,*

$$\left\| u(t) - \frac{\eta}{\sqrt{1+t}} V\left(\frac{\cdot}{\sqrt{1+t}}\right) \right\|_{L^q} \leq \frac{C_q}{(1+t)^{\frac{1}{2}+\mu-\frac{1}{q}}}, \tag{1.12}$$

où V est le champ de vecteur obtenu à partir de G par la loi de Biot-Savart.

La restriction $\mu < \frac{1}{2}$ vient du fait que la seconde valeur propre isolée de $\mathcal{L}$ est $-\frac{1}{2}$. Pour obtenir un meilleur taux de convergence, il est nécessaire de faire une étude des profils à l'ordre deux, ce qui revient à décomposer W sur les espaces propres associés aux deux premières valeurs propres de $\mathcal{L}$. Pour une étude à l'ordre 2, il faut de plus considérer un espace à poids polynomial de degré supérieur à 2, afin d'obtenir deux valeurs propres isolées dans le spectre de $\mathcal{L}$. Les profils asymptotiques à l'ordre 2 du système (1.2) sont également détaillés dans [36]. En 2005, T. Gallay et E. Wayne ont montré dans [38] que La condition de petitesse des théorèmes 1.1 et 1.2 sur les données initiales n'est pas nécessaire. De plus, il est montré que le taux optimal $\frac{1}{2}$ est atteint pour toute donnée dans $L^2(m)$, où $m > 2$. Dans un cadre fonctionnel différent, on peut aussi trouver des résultats similaires dans l'article plus ancien de Y. Giga et T. Kambe [42].

Profils asymptotiques en dimension 3

Considérons à présent un fluide newtonien remplissant tout l'espace $\mathbb{R}^3$. En posant $w = \text{rot}\, u = \nabla \wedge u$, les équations du tourbillon en dimension 3 sont données par

$$\begin{aligned} &\partial_t w - \Delta w + u.\nabla w - w.\nabla u = 0, \\ &\text{div}\, u = \text{div}\, w = 0, \\ &w_{|t=0} = w_0, \end{aligned} \tag{1.13}$$

où l'on a supposé $\nu = 1$ et où u est obtenu en appliquant la loi de Biot-Savart tridimensionnelle à w, donnée par

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \wedge w(y) dy. \tag{1.14}$$

Contrairement au cas de la dimension 2, w est un vecteur de $\mathbb{R}^3$ à divergence nulle. Ici encore, on travaille dans des espaces de Lebesgue à poids. Les solutions de (1.13) étant des vecteurs à divergence nulle, il est naturel de définir, pour $m \in \mathbb{N}$,

$$\mathbb{L}^2(m) = \left\{ u \in L^2(m)^3 : \text{div}\, u = 0 \right\},$$

où $L^2(m)$ est donné par (1.4), en remplaçant $\mathbb{R}^2$ par $\mathbb{R}^3$.

T. Gallay et E. Wayne ont montré en 2002 dans [37] que, pour peu que les données initiales soient petites dans un espace $L^2(m)$ bien choisi, les solutions de (1.13) convergent vers des champs de vecteurs de $\mathbb{R}^3$ dont chaque composante est une solution particulière des équations de la chaleur (1.5). Afin de préciser la nature de ces solutions limites, on pose

$$p_1(x) = \frac{1}{2}\begin{pmatrix} 0 \\ -x_2 \\ x_3 \end{pmatrix}, \quad p_2(x) = \frac{1}{2}\begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}, \quad p_3(x) = \frac{1}{2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix},$$

et on définit la fonction gaussienne

$$J(x) = \frac{1}{(4\pi)^{3/2}} e^{-\frac{|x|^2}{4}}.$$

Si w est une solution de (1.13) de donnée w_0 suffisamment petite dans $L^2(m)$, elle converge alors lorsque le temps tend vers l'infini vers

$$(t,x) \longmapsto \sum_{i=1}^{3} \frac{b_i}{(1+t)^2} f_i\left(\frac{x}{\sqrt{1+t}}\right), \tag{1.15}$$

où

$$f_i(x) = p_i(x)J(x), \tag{1.16}$$

et

$$b_i = \int_{\mathbb{R}^3} p_i(x).w_0(x).$$

A l'instar du cas de la dimension 2, ce résultat est obtenu via le changement de variables (1.8). En dimension 3, un rapide calcul montre que le champ de vecteur W défini par (1.8) satisfait l'égalité

$$\partial_\tau W - \mathcal{L}W + U.\nabla W - W.\nabla U = 0, \tag{1.17}$$

où $\mathcal{L}W = W + \Delta W + \frac{X}{2}.\nabla W$.

Une étude détaillée du spectre de $\mathcal{L}$ sur $\mathbb{L}^2(m)$, que l'on peut trouver dans [37], montre que celui-ci est la réunion du spectre discret

$$\sigma_d = \left\{-\frac{1}{2}(k+1) : k \in \mathbb{N}^*\right\}, \tag{1.18}$$

et du spectre continu

$$\sigma_c = \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \frac{1}{4} - \frac{m}{2} \right\}. \tag{1.19}$$

On constate que plus m est grand, et plus le spectre continu est décalé sur la gauche. Si m est assez grand, -1 est une valeur propre isolée de $\mathcal{L}$, de multiplicité 3, et la famille de vecteurs $\{f_1, f_2, f_3\}$ est une base de l'espace propre associé. On décompose ensuite W sur le spectre de $\mathcal{L}$, c'est-à-dire comme suit :

$$W(\tau) = \sum_{i=1}^{3} \beta_i(\tau) f_i + R(\tau),$$

où $\beta_i \in \mathbb{R}$ et $R(\tau)$ est un reste dans le supplémentaire $\mathcal{W}$ de l'espace propre associé à la valeur propre -1. Dans [37], il est montré, par une description de $\mathcal{W}$ et un calcul sur les solutions de (1.17), que $\beta_i(\tau) = e^{-\tau} \int_{\mathbb{R}^2} p_i(X).W\left(0, X\right) dX$. Dans [37], il est montré le théorème suivant.

Théorème 1.3 *Soit $1 < \mu \leq \frac{3}{2}$ une constante fixée et $m > 2\mu + \frac{1}{2}$. Il existe $r > 0$ telle que, pour toute donnée initiale $W_0 \in \mathbb{L}^2(m)$ avec $\|W_0\|_{L^2(m)} \leq r$, il existe une unique solution $W \in C^0\left(\mathbb{R}^+, \mathbb{L}^2(m)\right)$ de (1.13) de donnée initiale W_0 et une constante positive C, telles que, pour tout $\tau \geq 0$,*

$$\left\| W(t) - \sum_{i=1}^{3} e^{-\tau} b_i f_i \right\|_{L^2(m)} \leq C e^{-\mu\tau} \left\| w_0 \right\|_{L^2(m)}, \tag{1.20}$$

où $b_i = \int_{\mathbb{R}^3} p_i.W_0 dx$.

En revenant dans les variables de départ, on obtient le théorème suivant.

Théorème 1.4 *Soit $1 < \mu \leq \frac{3}{2}$ une constante fixée et $m > 2\mu + \frac{1}{2}$. Il existe $r > 0$ telle que, pour toute donnée initiale $w_0 \in \mathbb{L}^2(m)$ avec $\|w_0\|_{L^2(m)} \leq r$, il existe une unique solution $w \in C^0\left(\mathbb{R}^+, \mathbb{L}^2(m)\right)$ de (1.13). De plus, pour tout $2 \leq p \leq \infty$, il existe une constante positive C_p telle que, pour tout $t \geq 0$,*

$$\left\| w(t) - \sum_{i=1}^{3} \frac{b_i}{(1+t)^2} f_i \left(\frac{\cdot}{\sqrt{1+t}} \right) \right\|_{L^p} \leq C_p \left(1+t\right)^{-1-\mu-\frac{3}{2p}} \left\| w_0 \right\|_{L^2(m)}, \tag{1.21}$$

où $b_i = \displaystyle\int_{\mathbb{R}^3} p_i.w_0 dx$. Si u est le champ de vecteurs obtenu en appliquant la loi de Biot-Savart à w, alors, pour tout $2 \leq q \leq \infty$, il existe une constante positive C_q telle que, pour tout $t \geq 0$,

$$\left\| u(t) - \sum_{i=1}^{3} \frac{b_i}{(1+t)^{\frac{3}{2}}} v_i \left(\frac{\cdot}{\sqrt{1+t}} \right) \right\|_{L^q} \leq C_q \left(1+t\right)^{-\frac{1}{2}-\mu-\frac{3}{2q}} \|w_0\|_{L^2(m)}, \tag{1.22}$$

où v_i est obtenu en appliquant la loi de Biot-Savart (1.14) à f_i.

Comme pour le cas de la dimension 2, le taux optimal que l'on peut considérer pour une décomposition au premier ordre est lié à la seconde valeur propre du spectre discret de $\mathcal{L}$. Pour décrire les profils asymptotiques des solutions de (1.13) à un ordre supérieur, il est nécessaire de travailler dans des espaces à poids pour lesquels le spectre de $\mathcal{L}$ admet au moins deux valeurs propres isolées. On constate d'après (1.19), que prendre un poids polynomial grand permet de repousser la partie réelle du spectre continu de $\mathcal{L}$ sur la gauche, et d'obtenir ainsi des valeurs propres isolées. Le développement asymptotique des solutions de (1.13) est décrit jusqu'à l'ordre 2 dans [37], en travaillant dans un espace $\mathbb{L}^2(m)$, pour $m > \frac{7}{2}$.

1.2 Attracteurs

Dans le cas de nombreuses équations d'évolution autonomes, on peut définir un système dynamique ou semiflot $S(t)$ qui, à la donnée initiale u_0, associe la solution $u(t) \equiv S(t)u_0$ de l'équation considérée.

Définition 1.1 *Un système dynamique $S(t)$ (ou semi-groupe continu) sur un espace métrique X est une famille d'opérateurs continus $\{S(t) : X \to X, t \in \mathbb{R}^+\}$ satisfaisant les propriétés suivantes :*

(H1) $S(0) = Id$ et $S(t) \circ S(s) = S(t+s)$, pour tout $t, s \geq 0$.

(H2) Pour tout $t \in \mathbb{R}^+$, $S(t)$ est un opérateur continu sur X.

(H3) L'application $t \in \mathbb{R}^+ \to S(t)u$ est continue à valeurs dans X.

Si les propriétés ci-dessus sont vérifiées pour $t \in \mathbb{R}$, on parle de groupe continu.

Selon cette définition, si les solutions d'un système d'équations aux dérivées partielles sont associées à un système dynamique $S(t)$ sur un espace métrique X, $S(t)x$ est la solution de donnée initiale x prise au temps t. Afin de définir précisément ce qu'est un attracteur pour un système dynamique $S(t)$, nous définissons la semi-distance sur les sous-ensembles d'un espace métrique (X, d) comme suit :

$$\delta_X(A, B) = \sup_{b \in B} \inf_{a \in A} d(a, b).$$

Nous avons également besoin de la définition suivante.

Définition 1.2 *Soit (X, d) un espace métrique et $S(t)$ un système dynamique sur X.*

1. *On dit que l'ensemble A attire l'ensemble B si $\lim_{t \to +\infty} \delta_X(S(t)B, A) = 0$.*
2. *On dit que A est invariant par $S(t)$ si $S(t)A = A$, pour tout $t \geq 0$.*
3. *On dit que A est un attracteur global de S, si A est compact, invariant par S et attire tous les bornés de X.*

Sur un ouvert borné Ω de $\mathbb{R}^2$ avec conditions de Dirichlet, O. Ladyzhenskaya a montré que les solutions des équations de Navier-Stokes dans $H = \left\{u \in L^2(\Omega)^2 : \operatorname{div} u = 0, u_{|\partial\Omega} = 0\right\}$ permettent de définir un système dynamique $S(t)$ sur H ; quand les données initiales sont petites (voir [49]). De plus, elle a montré que le système dynamique $S(t)$ régularise en temps fini, c'est à dire que pour tout $t > 0$, $S(t)$ envoie H dans $H \cap H^1(\Omega)^2$ et que, dans le cas de données petites, il admet un attracteur local sur H. Ce résultat a ensuite été amélioré (voir [2] ou [50]) en un résultat d'existence d'attracteur global sur H, au sens de la définition 1.2. Il est montré de plus dans [50] que la régularité de l'attracteur $\mathcal{A}$ n'est limitée que par la régularité de la force extérieure f et que $S(t)$ est un groupe continu sur $\mathcal{A}$. On a le théorème d'existence d'un attracteur global suivant.

Théorème 1.5 *Soit $f \in H$ et $\nu > 0$. La famille d'opérateurs $\{S(t) : H \to H\}$ qui à une donnée u_0 associent $u(t)$ la solution de (1.1) de donnée u_0 au temps t est un semi-groupe continu sur H. De plus, $S(t)$ admet un attracteur global compact donné par*

$$\mathcal{A} = \left\{ \bigcap_{t \geq 0} S(t)B : B \text{ borné de } H \right\}.$$

En dimension 3, si la force f est petite, on peut montrer l'existence d'un attracteur local (voir par exemple [17]). Citons aussi les travaux de C. Foias et R. Temam, qui ont montré l'existence d'un attracteur dans un sens faible sur un domaine périodique $\mathbb{T}^3$ de $\mathbb{R}^3$, pour des solutions à valeurs dans $L^2(\mathbb{T}^3)$ (voir [28] et [30]). Ce résultat montre l'existence d'un attracteur pour la topologie faible de L^2 (voir aussi [60]).

2 Fluides de grade 2

Cette section traite du comportement asymptotique des solutions des équations des fluides de grade 2. On commence par rappeler quelques résultats connus sur l'existence de solutions à ce système, puis rappellera ce que l'on sait sur le comportement asymptotique de ces solutions. Dans le cadre de cette thèse, deux résultats sont exposés en lien avec ce sujet. Les travaux exposés dans le chapitre 3 donnent une description au premier ordre des profils asymptotiques des solutions des équations des fluides de grade 2 en dimension 3. Le chapitre 4, bien que consacré à l'étude des fluides de grade 3, apporte quelques précisions sur le comportement des fluides de grade 2 en dimension 2. On considère donc, pour un champ de vecteur $u \in \mathbb{R}^d$, où $d = 1, 2$, le problème de Cauchy suivant

$$\begin{aligned} &\partial_t (u - \alpha\Delta u) - \nu\Delta u + \text{rot }(u - \alpha\Delta u) \wedge u + \nabla p = f, \\ &\text{div } u = 0, \\ &u_{|t=0} = u_0, \end{aligned} \tag{2.1}$$

où $\nu > 0$ est la viscosité, $\alpha \geq 0$, p est la pression et f est la force extérieure appliquée au fluide. Si la dimension de l'espace est 2, on fait la convention $u = (u_1, u_2, 0)$ et rot $u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$.

2.1 Résultats d'existence

Le premier résultat d'existence est celui obtenu par D. Cioranescu et O. El Hacène sur un ouvert borné Ω de $\mathbb{R}^2$ ou $\mathbb{R}^3$, en considérant des conditions de Dirichlet sur le bord de Ω, c'est-à-dire $u_{|\partial\Omega} = 0$. (voir [19]). Pour cette étude, ils ont considéré les espaces fonctionnels

$$V = \left\{ u \in H^1(\Omega)^d : \text{div } u = 0, u_{|\partial\Omega} = 0 \right\}, \tag{2.2}$$

et, pour $s \geq 1$,

$$V^s = V \cap H^s(\Omega)^d. \tag{2.3}$$

Dans cet article, les auteurs ont construit des solutions faibles dans l'espace fonctionnel V^3 en considérant une force f à valeurs dans V^1. En dimension 2, le résultat qu'ils obtiennent est global en temps, alors que dans le cas de la dimension 3, l'existence n'est que locale en temps. Dans les deux cas, les solutions sont uniques. En dimension 2, on a le théorème suivant.

Théorème 2.1 *Soit Ω un ouvert borné de $\mathbb{R}^2$ dont le bord est régulier. Soit T une constante positive donnée, $f \in L^2((0,T), V)$ et $u_0 \in V^3$ donnés, il existe une unique solution faible u au système (2.1) telle que*

$$u \in L^\infty((0,T), V^3) \quad \text{et} \quad \partial_t u \in L^\infty\left((0,T), (V^3)'\right),$$

où $(V^3)'$ est l'espace dual de V^3.

En dimension 3, l'énoncé est le même, mais la solution n'existe que sur un intervalle $[0,T^*]$, où $T^* \leq T$. Ce théorème est obtenu par une méthode de Galerkin, c'est-à-dire la construction d'une suite u_n de solutions approchées à valeurs dans des espaces fonctionnels de dimensions finies. Dans ce cas, u_n est à valeurs dans l'espace vectoriel engendré par les n fonctions propres du produit scalaire associé à l'opérateur rot $(Id - \alpha\Delta)$. En passant à la limite lorsque n tend vers l'infini, on montre ensuite que u_n converge vers un certain u, qui est l'unique solution faible de (2.1). En particulier, on peut voir que le théorème ci-dessus est vrai pour une force $f \in V$ ne dépendant pas du temps. En dimension 2, ce résultat a été amélioré par I. Moise, R. Rosa et X. Wang dans [53], qui ont montré que les solutions données par ce théorème sont continues en temps. En dimension 2 ou 3, G. Galdi, M. Grobbelaar Van Dalsen et N. Sauer on montré l'existence de solutions fortes dans l'espace V^m, avec $m \geq 5$, qui sont de plus globales si les données initiales sont supposées petites dans V^m (voir [35]). Finalement, D. Cioranescu et V. Girault ont montré dans [18] le théorème suivant.

Théorème 2.2 *Soit Ω un ouvert borné de $\mathbb{R}^3$ dont le bord est régulier et $m \geq 3$. Soit $f \in L^2(\mathbb{R}^+, V^{m-2})$ et $u_0 \in V^m$ donnés, il existe $T^* \in [0,+\infty]$ et une unique solution faible u au système (2.1) telle que*

$$u \in L^\infty\left((0,T^*), V^m\right) \quad \text{et} \quad \partial_t u \in L^\infty\left((0,T^*), V^{m-2}\right),$$

De plus, il existe $R > 0$ tel que, si $\|u_0\|_{H^3} \leq R$, alors $T^* = +\infty$. Si $m \geq 4$, alors u est une solution classique de (2.1).

En dimension 3, une extension de ces deux théorème est donnée par D. Bresch et J. Lemoine dans [8]. En effet, pour une force f à valeurs dans $L^r(\Omega)^3$ avec $r > 3$, les auteurs ont montré l'existence de solutions locales dans l'espace $V^1 \cap W^{2,r}(\Omega)^3$. Plus précisément, ils ont démontré le théorème d'existence suivant.

Théorème 2.3 *Soit Ω un ouvert borné de $\mathbb{R}^3$, $\alpha > 0$ et $f \in L^r\left([0,T] \times \Omega\right)^3$, avec $r > 3$. Il existe T^*, $0 < T^* \leq T$ et une unique solution au système (2.1) tels que*

$$u \in C^0\left([0,T^*], V^1 \cap W^{2,r}(\Omega)^3\right) \quad \text{et} \quad \partial_t u \in L^\infty\left([0,T^*], V^1 \cap W^{1,r}(\Omega)^3\right).$$

Si de plus $f \in L^\infty\left(\mathbb{R}^+, L^r(\Omega)^3\right)$, il existe trois constantes positives $r_1 = r_1(r,\Omega)$, $r_2 = r_2(\alpha,\nu,r,\Omega)$ et $r_3 = r_3(\alpha,\nu,r,\Omega)$ telles que si

$$\begin{cases} \alpha \geq r_1, \\ \|f\|_{L^\infty(\mathbb{R}^+, L^r(\Omega)^3)} \leq r_2, \\ \|u_0 - \Delta u_0\|_{L^r} \leq r_3, \end{cases}$$

alors $T^* = +\infty$ *et l'on a*

$$u \in C_b^0\left(\mathbb{R}^+, V^1 \cap W^{2,r}(\Omega)^3\right) \quad et \quad \partial_t u \in L^\infty\left(\mathbb{R}^+, V^1 \cap W^{1,r}(\Omega)^3\right).$$

Le démonstration de ce théorème repose sur un théorème de point fixe de Schauder. On peut trouver d'autres résultats d'existence pour les équations des fluides de grade 2 dans [5], [7], [18], [20], [21], [34], [52] ou [54].

2.2 Dynamique des fluides de grade 2

On rappelle dans cette section quelques résultats sur le comportement asymptotique des solutions des équations des fluides de grade 2. Pour ces équations, il existe des résultats qui concernent l'existence d'un attracteur (voir [53], [57] ou [56]) et un article traitant des profils asymptotiques des solutions de ces équations (voir [47]).

Profils asymptotiques

Les profils asymptotiques des solutions des équations des fluides de grade 2 ont été étudiés en dimension 2 par B. Jaffal-Mourtada dans [47] pour un fluide de grade 2 remplissant tout l'espace $\mathbb{R}^2$. Comme pour le cas des équations de Navier-Stokes, le système auquel on s'intéresse ici est celui satisfait par le tourbillon $w = \operatorname{rot} u$. En dimension 2, ces équations sont données par

$$\begin{aligned} &\partial_t \left(w - \alpha\Delta w\right) - \Delta w + u.\nabla\left(w - \alpha\Delta w\right) = 0, \\ &w_{|t=0} = w_0, \end{aligned} \tag{2.4}$$

où u est reconstitué par la loi de Biot-Savart (1.3).

Dans [47], il est montré que les profils asymptotiques des solutions des équations des fluides de grade 2 sont les mêmes que ceux décrits par T. Gallay et E. Wayne (voir [36]). En effet, les solutions des équations des fluides de grade 2 convergent elles aussi vers les solutions autosimilaires des équations de la chaleur données par (1.6). Pour cette étude, l'auteur a été amenée à considérer des espaces de Sobolev d'ordre 2 à poids, définis, pour $m \in \mathbb{N}$, par

$$H^2(m) = \left\{u \in L^2(m) : \partial^n u \in L^2(m), n \in \mathbb{N}^2, |n| \leq 2\right\}.$$

Comme pour le cas des équations de Navier-Stokes, ce résultat s'obtient en considérant des variables autosimilaires comparables à celles définies par (1.8), mais en prenant $X =$

$\dfrac{x}{\sqrt{t+T}}$ et $\tau = \ln(t+T)$, pour une constante positive T fixée. Cette constante T est introduite pour pouvoir traiter le problème indépendamment de la taille de α. En posant

$$\begin{aligned} w(t,x) &= \frac{1}{t+T} W\left(\ln(t+T), \frac{x}{\sqrt{t+T}}\right), \\ u(t,x) &= \frac{1}{\sqrt{t+T}} U\left(\ln(t+T), \frac{x}{\sqrt{t+T}}\right), \end{aligned} \tag{2.5}$$

un calcul simple montre que W satisfait le système d'équations

$$\begin{aligned} &\partial_\tau \left(W - \alpha e^{-\tau}\Delta W\right) - \mathcal{L}(W) + U.\nabla\left(W - \alpha e^{-\tau}\Delta W\right) \\ &\qquad\qquad\qquad + \alpha e^{-\tau}\Delta W + \alpha e^{-\tau}\tfrac{X}{2}.\nabla\Delta W = 0, \\ &W_{|\tau=\ln(T)} = W_0, \end{aligned} \tag{2.6}$$

où $W_0(X) = T w_0\left(\sqrt{T}X\right)$et $\mathcal{L}(W) = W + \Delta W + \frac{X}{2}.\nabla W$.

Le système (2.6) est initialisé au temps $\tau = \ln(T)$, qui est arbitraire puisque T l'est. En prenant T suffisamment grand, on peut donc considérer $\alpha e^{-\tau}$ aussi petit que l'on veut. Comme pour le cas des fluides newtoniens, on considère la décomposition

$$W(\tau) = \eta G + R,$$

où $\eta = \displaystyle\int_{R^2} W_0(X)dX = \int_{R^2} w_0(x)dx$ et G est le tourbillon d'Oseen donné par (1.7), qui est un vecteur propre de $\mathcal{L}$ associé à la valeur propre 0.

En effectuant des estimations d'énergies sur R dans divers espaces fonctionnels, dont des espaces de Sobolev à poids, B. Jaffal-Mourtada a montré le théorème suivant.

Théorème 2.4 *Il existe trois constantes positives T, θ, $0 < \theta < \frac{1}{2}$ et $\gamma = \gamma(T)$ telles que, pour tout $W_0 \in H^2(2)$ avec $\|W_0\|_{H^2(2)} \leq \gamma$, il existe une unique solution $W \in C^0\left([\ln(T), +\infty), H^2(2)\right)$ au système (2.6) telle que,*

$$\left\|\left(1 - \alpha e^{-\tau}\Delta\right)\left(W(\tau) - \eta G\right)\right\|_{L^2(2)} \leq C e^{-\frac{\theta}{2}\tau}, \tag{2.7}$$

où $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(X)dX = \int_{\mathbb{R}^2} w_0(x)dx$, C est une constante positive et G est le tourbillon d'Oseen, donné par (1.7).

En revenant dans les variables de départ, on obtient le théorème suivant, qui est un corollaire du théorème 2.4.

Théorème 2.5 *Il existe trois constantes positives T, θ, $0 < \theta < \frac{1}{2}$ et $\gamma = \gamma(T)$ telles que, pour tout $w_0 \in H^2(2)$ avec $\|w_0\|_{H^2(2)} \leq \gamma$, il existe une unique solution $w \in C^0(\mathbb{R}^+, H^2(2))$ au système (2.4) telle que, pour tout $1 \leq p \leq 2$,*

$$\left\| (1-\alpha\Delta)\, w(t) - \frac{\eta}{t+T} G\left(\frac{\cdot}{\sqrt{t+T}}\right) \right\|_{L^p} \leq C_p \,(t+T)^{-1-\frac{\theta}{2}+\frac{1}{p}}\,, \tag{2.8}$$

où C_p est une constante positive, $\eta = \int_{\mathbb{R}^2} w_0(x)dx$ et G est donné par (1.7). Si u est le champ de vitesses obtenu en appliquant la loi de Biot-Savart (1.3) à w, alors, pour tout $1 \leq q \leq 2$,

$$\left\| (1-\alpha\Delta)\, u(t) - \frac{\eta}{t+T} V\left(\frac{\cdot}{\sqrt{t+T}}\right) \right\|_{L^q} \leq C_q \,(t+T)^{-\frac{1}{2}-\frac{\theta}{2}+\frac{1}{q}}\,, \tag{2.9}$$

où C_q est une constante positive et V est le champ de vitesses obtenu en appliquant la loi de Biot-Savart (1.3) à G.

Il est à noter que le taux de convergence n'est ici pas optimal, et est nécessairement plus petit que $\frac{1}{4}$. Pour les raisons spectrales expliquées précédemment, le taux optimal est $\frac{1}{2}$. Dans le chapitre 4, qui porte sur les fluides de grade 3, on donnera une amélioration de ce résultat, en montrant que l'on peut prendre un taux de convergence aussi proche du taux optimal que souhaité, pourvu que les données initiales soient suffisamment petites dans $H^2(2)$. D'un point de vue concret, en comparant ce résultat avec les théorèmes 1.1 et 1.2, ce résultat montre que les fluides de grade 2 se comportent asymptotiquement comme les fluides newtoniens.

Attracteurs en dimension 2

Sur un domaine borné de $\mathbb{R}^2$, l'existence d'un attracteur global pour les équations des fluides de grade 2 (2.1) est connue dans l'espace fonctionnel V^3. Les résultats que l'on présente ici sont vrais également pour un domaine périodique $\mathbb{T}^2 = [-\pi,\pi]^2$ de $\mathbb{R}^2$, en définissant

$$H_{per} = \left\{ u \in L^2(\mathbb{T}^2)^2 : \operatorname{div} u = 0, \textstyle\int_{\mathbb{T}^2} u dx = 0 \right\},$$

et en remplaçant V^s par

$$V^s_{per} = H_{per} \cap H^s(\mathbb{T}^2)^2.$$

Sur un domaine borné ou périodique de $\mathbb{R}^2$, le théorème 2.1 assure l'existence de solutions globales dans V^3 au système (2.1) pour une force $f \in V$ indépendante du temps. Le théorème suivant, démontré par I. Moise, R. Rosa et X. Wang dans [53], montre l'existence d'un attracteur pour les équations des fluides de grade 2 sur V^3.

Théorème 2.6 *Soit $\Omega \in \mathbb{R}^2$ un ouvert borné, simplement connexe de bord connexe régulier, et soient $\nu > 0$, $\alpha > 0$ et $f \in H^1(\Omega)^2$. Si $S(t)u_0 = u(t)$ est l'unique solution de (2.1) de donnée initiale u_0, alors $S(t)$ est un groupe continu sur V^3. De plus, $S(t)$ possède un attracteur global dans V^3.*

La méthode utilisée pour obtenir ce résultat s'appuie sur des égalités d'énergies satisfaites par les solutions de (2.1). Une partie des résultats du chapitre 4 s'appuie sur cette méthode. Dans le cas des conditions périodiques, il est de plus montré dans [57] que si la force est à valeurs dans $V_{per} \cap H^{1+m}(\mathbb{T}^2)^2$, où m est une constante positive, alors il existe une constante positive δ tel que l'attracteur $\mathcal{A}$ appartient à l'espace $V^3_{per} \cap H^{3+\delta}(\mathbb{T}^2)^2$.

Sur un domaine périodique, M. Paicu, G. Raugel ont étendu ce résultat à l'espace de Sobolev $W^{3,p}(\mathbb{T}^2)^2$, avec $1 < p < \infty$, où $W^{s,q}$ désigne l'espace de Sobolev d'ordre s associé à la norme L^q (voir [56]). La méthode utilisée dans ce cas est différente de celle de [53], et repose entre autres sur un changement de variables en coordonnées lagrangiennes.

2.3 Contributions de la thèse

Dans le cadre de cette thèse, une description des profils asymptotiques à l'ordre 1 pour les équations des fluides de grade 2 sur $\mathbb{R}^3$ est donnée dans le chapitre 3. En reprenant la méthode utilisée par B. Jaffal-Mourtada pour montrer les théorèmes 2.4 et 2.5, on obtient un résultat similaire en dimension 3, qui montre que les solutions des équations des fluides de grade 3 convergent vers les solutions autosimilaires des équations de la chaleur données par (1.15). La conclusion que l'on peut en faire est qu'en dimension 3, le comportement asymptotique des fluides de grade 2 est comparable à celui des fluides newtoniens. Le système que l'on considère pour ce travail est celui satisfait pour le tourbillon $w = \operatorname{rot} u$. En dimension 3, ce système est donné par

$$\begin{aligned} &\partial_t (w - \alpha\Delta w) - \Delta w + u.\nabla (w - \alpha\Delta w) - (w - \alpha\Delta w).\nabla u = 0, \\ &\operatorname{div} w = \operatorname{div} u = 0, \\ &w_{|t=0} = w_0, \end{aligned} \tag{2.10}$$

où u est reconstitué par la loi de Biot-Savart tridimensionnelle (1.14).

Afin d'énoncer le théorème démontré dans le chapitre 3, on définit l'espace de fonctions dans lequel les solutions sont définies. Pour $m \in \mathbb{N}$, on pose

$$\mathbb{H}^2(m) = \left\{ u \in \mathbb{L}^2(m) : \partial^\alpha u \in \mathbb{L}^2(m), |\alpha| \leq 2 \right\}.$$

Le résultat que l'on obtient se montre de la même façon que le théorème 2.4. En considérant le changement de variables $X = \dfrac{x}{\sqrt{t+T}}$ et $\tau = \ln(t+T)$ et en définissant W par la relation (2.5), on obtient le système

$$\begin{aligned}
&\partial_\tau \left(W - \alpha e^{-\tau}\Delta W\right) - \mathcal{L}(W) + U.\nabla\left(W - \alpha e^{-\tau}\Delta W\right) - \left(W - \alpha e^{-\tau}\Delta W\right).\nabla U \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \alpha e^{-\tau}\Delta W + \alpha e^{-\tau}\tfrac{X}{2}.\nabla\Delta W = 0, \\
&\operatorname{div} W = \operatorname{div} U = 0, \\
&W_{|\tau=\ln(T)} = W_0,
\end{aligned} \tag{2.11}$$

où $W_0(X) = Tw_0(\sqrt{T}X)$ et $\mathcal{L}(W) = W + \Delta W + \frac{X}{2}.\nabla W$.

Encore une fois, on décompose W sur le spectre de $\mathcal{L}$. En dimension 3, la première valeur propre de $\mathcal{L}$ est -1, et une base de l'espace propre associé est $\{f_1, f_2, f_3\}$, donnée par (1.16). On décompose W comme suit :

$$W(\tau) = e^{-\tau}\sum_{i=1}^{3} b_i f_i + R(\tau),$$

où $p_i = \displaystyle\int_{\mathbb{R}^3} p_i(X).W(\tau,X)dX = \int_{\mathbb{R}^3} p_i(X).W_0(X)dX$.

En effectuant notamment des estimations d'énergies dans divers espaces fonctionnels, on montrera dans le chapitre 3 le théorème suivant.

Théorème 2.7 *Soit θ, $0 < \theta < \frac{3}{2}$ une constante positive fixée et $W_0 \in \mathbb{H}^2(4)$. Il existe $\gamma_0 = \gamma_0(\alpha) > 0$ et $T_0 = T_0(\alpha) \geq 1$ tels que, si $T \geq T_0$ et s'il existe $\gamma \leq \gamma_0$ tel que $W_0 \in \mathbb{H}^2(4)$ satisfait l'inégalité*

$$\|W_0\|^2_{L^2(4)} + \|\nabla W_0\|^2_{L^2} + \alpha e^{-\tau_0}\|\Delta W_0\|^2_{L^2} + \alpha^2 e^{-2\tau_0}\left\||X|^4\Delta W_0\right\|^2_{L^2} \leq \gamma\left(\frac{3}{2} - \theta\right)^2, \tag{2.12}$$

où $\tau_0 = \log(T)$,

alors il existe une unique solution $W \in C^0\left([\tau_0, +\infty), \mathbb{H}^2(4)\right)$ au système (2.11) et une *constante positive $C = C(\theta, \alpha, \gamma, T_0)$ tels que*

$$\left\| \left(Id - \alpha e^{-\tau}\Delta\right)\left(W(\tau) - e^{-\tau}\sum_{i=1}^{3} b_i f_i\right)\right\|_{L^2(4)} \leq Ce^{-\theta\tau}, \tag{2.13}$$

où $b_i = \int_{\mathbb{R}^3} p_i(X).W_0(X)dX$.

L'espace $\mathbb{H}^2(4)$ dans lequel on travaille est choisi pour pouvoir considérer un taux de convergence aussi proche que souhaité du taux optimal. Comme expliqué plus tôt, le taux de convergence que l'on obtient est lié à la seconde valeur propre discrète du spectre de $\mathcal{L}$ sur l'espace $\mathbb{L}^2(m)$. Dans notre cas, cette valeur propre est $-\frac{3}{2}$ et on choisi donc m suffisamment grand pour que la partie réelle du spectre continu soit repoussée au delà de $-\frac{3}{2}$. D'après (1.18), le m minimal que l'on doit prendre est donc $\frac{7}{2}$. Pour des raisons pratiques, on préfère travailler dans $\mathbb{L}^2(4)$. Dans les variables de départ, le théorème 2.7 donne le résultat suivant.

Théorème 2.8 *Soit θ, $0 < \theta < \frac{3}{2}$ une constante positive fixée et $w_0 \in \mathbb{H}^2(4)$. Il existe $\gamma_0 = \gamma_0(\alpha) > 0$ et $T_0 = T_0(\alpha) \geq 1$ tels que, pour tout $T \geq T_0$, $0 < \gamma \leq \gamma_0$ et w_0 satisfaisant la condition*

$$\begin{aligned} T^{1/2}\left\|w_0\right\|_{L^2}^2 + T^{-7/2}\left\||x|^4 w_0\right\|_{L^2}^2 + T^{3/2}\left\|\nabla w_0\right\|_{L^2}^2 \\ +\alpha T^{3/2}\left\|\Delta w_0\right\|_{L^2}^2 + \alpha^2 T^{-3/2}\left\||x|^4\Delta w_0\right\|_{L^2}^2 \leq \gamma\left(\tfrac{3}{2}-\theta\right)^2, \end{aligned} \tag{2.14}$$

il existe une unique solution $w \in C^0\left([0, +\infty), \mathbb{H}^2(4)\right)$ au système (2.10) et, pour tout $1 \leq p \leq 2$, l'inégalité suivante est satisfaite :

$$\left\|(Id - \alpha\Delta)\left(w(t) - \sum_{i=1}^{3}\frac{b_i}{(t+T)^2} f_i\left(\frac{\cdot}{\sqrt{t+T}}\right)\right)\right\|_{L^p} \leq C\,(t+T)^{-1-\theta+\frac{3}{2p}}, \tag{2.15}$$

où $C = C(\theta, \alpha, \gamma, T_0, p)$ est une constante positive.

De plus, il existe une constante positive $C = C(\theta, \alpha, \gamma, T_0)$ telle que

$$\left\||x|^4 (Id - \alpha\Delta)\left(w(t) - \sum_{i=1}^{3}\frac{b_i}{(t+T)^2} f_i\left(\frac{\cdot}{\sqrt{t+T}}\right)\right)\right\|_{L^2} \leq C\,(t+T)^{\frac{1}{4}-\theta}, \tag{2.16}$$

où $b_i = \frac{1}{T}\int_{\mathbb{R}^3} p_i(x).w_0(x)dx$ et f_i est donné par (1.15).

En particulier, en comparant avec le théorème 1.4, on constate que l'on obtient le même taux de convergence dans les espaces de Lebesgue classiques que celui obtenu pour les équations de Navier-Stokes.

3 Fluides de grade 3

Une partie significative de cette thèse traite du comportement asymptotique des solutions des équations des fluides de grade 3. La littérature mathématique consacrée à cette classe de fluides est plus restreinte que celle consacrée aux fluides de grade 2, et sans comparaison par rapport à celle consacrée aux équations de Navier-Stokes. S'il existe plusieurs théorèmes d'existence de solutions, l'asymptotique n'est que très peu étudiée. Dans cette section, on considère le système d'équations des fluides de grade 3, donné par

$$\begin{aligned}
&\partial_t (u - \alpha_1 \Delta u) - \nu \Delta u + \mathrm{rot}\ (u - \alpha_1 \Delta u) \wedge u - (\alpha_1 + \alpha_2) \left(A \Delta u + 2\mathrm{div}\ (L^t L)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\beta \mathrm{div}\ \left(|A|^2 A\right) + \nabla p = f, \\
&\mathrm{div}\, u = 0, \\
&u_{|t=0} = u_0.
\end{aligned} \tag{3.1}$$

où $u \in \mathbb{R}^d$, $\nu \geq 0$ est la viscosité $\alpha_1 \geq 0$, $\alpha_2 \in \mathbb{R}$, $\beta \geq 0$, p est la pression du fluide et f la force extérieure appliquée au fluide.

3.1 Résultats d'existence

Sur un ouvert borné régulier Ω de $\mathbb{R}^2$ ou $\mathbb{R}^3$, C. Amrouche et D. Cioranescu ont montré en 1997 l'existence locale de solutions pour des données dans l'espace V^3, défini par (2.3). Pour une donnée $u_0 \in V^3$ et une force $f \in L^2([0,T], V^1)$, ils ont montré qu'il existe un temps positif T^*, $0 < T^* \leq T$ et une solution u au système (3.1) tels que

$$u \in L^\infty([0,T^*], V^3) \quad \text{et} \quad \partial_t u \in L^2([0,T^*], V^1).$$

Pour ce travail, qui s'appuie sur un schéma de Galerkin, les auteurs on supposé la condition suivante, justifiée par des considérations physiques :

$$|\alpha_1 + \alpha_2| \leq \sqrt{24 \nu \beta}. \tag{3.2}$$

Les solutions obtenues sont de plus uniques (voir [1]). En dimension 3, une classe plus générale de solutions a été introduite par D. Bresch et J. Lemoine dans [9], à savoir des solutions dont les données initiales appartiennent à l'espace $V^1 \cap W^{2,r}(\Omega)^3$, où $r > 3$, pour une force extérieure à valeurs dans $W^{1,r}(\Omega)^3$. En utilisant le théorème de point fixe de Schauder, les auteurs ont montré dans ce cas l'existence de solutions locales aux équations des fluides de grade 3 dans l'espace $C^0([0,T], V^1 \cap W^{2,r}(\Omega)^3)$. De plus, les solutions obtenues sont uniques et les auteurs ne considèrent pas la restriction (3.2).

Il est également montré dans [9] que les solutions sont globales sous des conditions de petitesse sur les données initiales et la force extérieure. Il existe aussi d'autres résultats d'existence sur un ouvert borné (voir [6], [11] ou [61]).

Dans le cas d'un fluide de grade 3 remplissant tout l'espace $\mathbb{R}^2$ ou $\mathbb{R}^3$, V. Busuioc et D. Iftimie ont montré dans [10] en 2004 l'existence de solutions faibles globales dans l'espace de Sobolev $H^2(\mathbb{R}^d)$, pour une force $f \in L^\infty_{loc}\left(\mathbb{R}^+, L^2(\mathbb{R}^d)^d\right)$. Plus précisément, ils ont démontré le théorème suivant.

Théorème 3.1 *Considérons l'équation (3.1) sur $\mathbb{R}^d$, $d = 2, 3$, avec $f \in L^\infty_{loc}\left(\mathbb{R}^+, L^2(\mathbb{R}^d)^d\right)$, $u_0 \in H^2(\mathbb{R}^d)^d$, $div\, u_0 = 0$. Il existe une solution globale u telle que $u \in C^0_w\left(\mathbb{R}^+, H^2(\mathbb{R}^d)^d\right) \cap C^0\left(\mathbb{R}^+, H^s(\mathbb{R}^d)^d\right)$, pour tout $0 \leq s < 2$. De plus, si $d = 2$, cette solution est unique.*

Ce théorème s'obtient par un schéma de Friedrich et des estimations a priori dans $H^2(\mathbb{R}^d)$. Le même théorème peut être démontré sur un tore de $\mathbb{R}^2$ ou $\mathbb{R}^3$. En 2008, une nouvelle classe plus générale de solutions a été introduite par M. Paicu dans [55]. En reprenant une partie des résultats de [10] et en considérant des restrictions sur la taille des paramètres α_1 et α_2, il a démontré l'existence de solutions faibles globales dans l'espace $H^1(\mathbb{R}^d)$, lorsque la force f est à valeurs dans $H^{-1} + W^{-1,\frac{4}{3}}$, où $W^{-1,\frac{4}{3}}$ est le dual de l'espace de Sobolev $W^{1,4}$. De plus, ces solutions satisfont une égalité d'énergie dont on se servira dans le chapitre 4. Le théorème obtenu par M. Paicu est le suivant.

Théorème 3.2 *Soit $u_0 \in H^1(\mathbb{R}^d)^d$ tel que $div\, u_0 = 0$ et $f = f_1 + f_2$, où $f_1 \in L^\infty_{loc}\left(\mathbb{R}^+, H^{-1}(\mathbb{R}^d)^d\right)$ et $f_2 \in L^{4/3}_{loc}\left(\mathbb{R}^+, W^{-1,\frac{4}{3}}(\mathbb{R}^d)^d\right)$. Si les paramètres α_1, α_2, ν et β satisfont les conditions suivantes :*
- *$d = 2$: $\nu \geq 0$, $\beta > 0$ et $|\alpha_1| \leq \sqrt{8\nu\beta}$,*
- *$d = 3$: $\nu \geq 0$, $\beta > 0$, $3\alpha_1^2 + 4\left(\alpha_1 + \alpha_2\right)^2 \leq 24\nu\beta$,*

alors il existe une solution faible globale u au système (3.1), telle que

$$u \in C^0\left(\mathbb{R}^+, H^1(\mathbb{R}^d)^d\right) \cap L^4_{loc}\left(\mathbb{R}^+, W^{1,4}(\mathbb{R}^d)^d\right).$$

Dans le théorème ci-dessus, l'unicité n'est pas connue en général, et on verra plus tard que cela jouera un rôle important pour montrer l'existence d'un attracteur pour les solutions H^1. En effet, on ne pourra pas associer les solutions des équations des fluides de grade 3 données dans ce théorème à un système dynamique au sens classique.

3.2 Contributions de la thèse

Dans le cadre de cette thèse, deux résultats nouveaux sur les équations des fluides de grade 3 sont démontrés. Le premier concerne les profils asymptotiques des solutions de ces équations, pour un fluide remplissant tout l'espace $\mathbb{R}^2$. Le second montre l'existence d'un attracteur dans la topologie H^1 pour un fluide défini sur un tore de $\mathbb{R}^2$, pour des solutions plus régulières que H^1.

Profils asymptotiques sur $\mathbb{R}^2$

Dans le chapitre 4, on donne une description des profils asymptotiques au premier ordre des solutions des équations des fluides de grade 3 considérés sur l'espace entier $\mathbb{R}^3$. On montre, à l'instar des fluides newtoniens et de grade 2, que les solutions des équations des fluides de grade 3 convergent lorsque le temps tend vers l'infini vers les solutions autosimilaires des équations de la chaleur données par (1.6). Ce résultat est obtenu sous une condition de petitesse sur les données initiales, que l'on retrouve aussi pour le cas des fluides de grade 2 mais pas pour les équations de Navier-Stokes. Les équations que l'on considère pour ce travail sont les équations satisfaites par le tourbillon $w = \text{rot } u = \partial_1 u_2 - \partial_2 u_1$, données par

$$\begin{aligned} &\partial_t (w - \alpha\Delta w) - \Delta w + u.\nabla (w - \alpha\Delta w) - \beta \text{rot div } \left(|A|^2 A\right) = 0, \\ &w_{|t=0} = w_0, \end{aligned} \tag{3.3}$$

où u est reconstitué par la loi de Biot-Savart bidimensionnelle (1.3).

Remarquons que la constante α_2 n'apparait plus dans ce système d'équations. En effet, en dimension 2, on peut montrer que le terme avec la constante multiplicative α_2 du système (3.1) est un terme gradient qui par conséquent n'intervient pas dans le comportement des solutions. Ce phénomène est très spécifique à la dimension 2 et n'existe pas en dimension 3. On peut d'ailleurs constater que les conditions sur les paramètres du théorème 3.2 en dimension 2 ne portent pas sur α_2. Par conséquent, pour l'étude des profils asymptotiques, la différence entre les équations des fluides de grade 3 et les équations des fluides de grade 2 vient du terme cubique $\beta \text{div } \left(|A|^2 A\right)$.

Comme pour le cas des fluides newtoniens ou de grade 2, pour étudier les profils asymptotiques des solutions de (3.3), on considère les variables d'échelles. Pour T fixé, on définit W par le changement de variables (2.5) et, par un calcul simple, on montre que

W satisfait le système d'équations

$$\begin{aligned} &\partial_\tau \left(W - \alpha e^{-\tau} \Delta W\right) - \mathcal{L}(W) + U.\nabla \left(W - \alpha e^{-\tau} \Delta W\right) \\ &\qquad\qquad\qquad\qquad + \alpha e^{-\tau} \Delta W + \alpha e^{-\tau} \tfrac{X}{2}.\nabla \Delta W - \beta e^{-2\tau} \mathrm{rot\ div}\ \left(|A|^2 A\right) = 0, \\ &W_{|\tau = \ln(T)} = W_0, \end{aligned} \tag{3.4}$$

où $W_0(X) = T w_0(\sqrt{T} X)$ et $\mathcal{L}(W) = W + \Delta W + \frac{X}{2}.\nabla W$.

De la même manière que pour les fluides de grade 2, en décomposant W sur le spectre de $\mathcal{L}$ et en effectuant des estimations d'énergies dans divers espaces fonctionnels, on montrera le théorème suivant.

Théorème 3.3 *Soit θ, $0 < \theta < \dfrac{1}{2}$ une constante positive. Il existe deux constantes positives $\gamma_0 = \gamma_0(\alpha_1, \beta)$ et $T_0 = T_0(\alpha_1, \beta) \geq 1$ telles que, pour tout $T \geq T_0$, $0 < \gamma \leq \gamma_0$ et $W_0 \in H^2(2)$ satisfaisant la condition*

$$\|W_0\|_{H^1}^2 + \alpha_1 e^{-\tau_0} \|\Delta W_0\|_{L^2}^2 + \left\| |X|^2 W_0 \right\|_{L^2}^2 + \alpha_1^2 e^{-2\tau_0} \left\| |X|^2 \Delta W_0 \right\|_{L^2}^2 \leq \gamma \left(\frac{1}{2} - \theta \right)^6, \tag{3.5}$$

où $\tau_0 = \ln(T)$,

il existe une unique solution $W \in C^0\left([\tau_0, +\infty), H^2(2)\right)$ au système (2.6) et une constante positive $C = C(\alpha_1, \beta, \theta, \gamma)$ telles que, pour tout $\tau \geq \tau_0$,

$$\left\| \left(1 - \alpha_1 e^{-\tau} \Delta\right) \left(W(\tau) - \eta G\right) \right\|_{L^2(2)} \leq C e^{-\theta \tau}, \tag{3.6}$$

où $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(X) dX$.

Dans ce théorème, remarquons que l'on peut choisir le taux de convergence aussi proche du taux optimal $-\frac{1}{2}$ que souhaité, à condition de considérer des données suffisamment petites dans $H^2(2)$. Étant donné que les fluides de grade 2 sont un cas particulier des fluides de grade 3, le théorème 3.3 apporte donc une amélioration au théorème 2.4, où le taux de convergence considéré était au mieux $\frac{1}{4}$. Dans les variables de départ, le théorème 3.3 donne le résultat suivant.

Théorème 3.4 *Soit θ, $0 < \theta < \frac{1}{2}$ une constante positive. Il existe deux constantes positives $\gamma_0 = \gamma_0(\alpha_1, \beta)$ et $T_0 = T_0(\alpha_1, \beta) \geq 1$ telles que, pour tout $T \geq T_0$, $0 < \gamma \leq \gamma_0$*

et $w_0 \in H^2(2)$ *satisfaisant la condition*

$$T\left\|w_0\right\|_{L^2}^2+T^2\left\|\nabla w_0\right\|_{L^2}^2+\frac{1}{T}\left\||x|^2 w_0\right\|_{L^2}^2+\alpha_1 T^3\left\|\Delta w_0\right\|_{L^2}^2+\frac{\alpha_1^2}{T}\left\||x|^2 \Delta w_0\right\|_{L^2}^2 \leq \gamma\left(\frac{1}{2}-\theta\right)^6, \tag{3.7}$$

il existe une unique solution $w \in C^0\left([0,+\infty), H^2(2)\right)$ *de (3.3) telle que, pour tout* $1 \leq p \leq 2$ *et tout* $t \geq 0$,

$$\left\|(1-\alpha_1 \Delta)\left(w(t)-\frac{\eta}{t+T} G\left(\frac{\cdot}{\sqrt{t+T}}\right)\right)\right\|_{L^p} \leq C\,(t+T)^{-1+\frac{1}{p}-\theta},$$

où $C=C(p, \alpha_1, \beta, \theta, \gamma)$ *est une constante positive,*

et il existe une constante positive $C=C(\alpha_1, \beta, \theta, \gamma)$ *telle que, pour tout* $t \geq 0$,

$$\left\||x|^2(1-\alpha_1 \Delta)\left(w(t)-\frac{\eta}{t+T} G\left(\frac{\cdot}{\sqrt{t+T}}\right)\right)\right\|_{L^2} \leq C\,(t+T)^{\frac{1}{2}-\theta},$$

où $\eta=\int_{\mathbb{R}^2} w_0(x) dx$ *et* G *est le vortex d'Oseen, donné par (1.7).*

Comme pour le cas des fluides de grade 2, on peut là aussi conclure que les fluides de grade 3 se comportent asymptotiquement comme les fluides newtoniens régis par les équations de Navier-Stokes, du moins au premier ordre.

Attracteur

Il n'existe pas aujourd'hui de résultat traitant de l'existence d'un attracteur pour les équations des fluides de grade 3. Dans le chapitre 4, on montre l'existence d'un attracteur dans un sens plus faible que celui de la définition 1.2, pour un fluide sur le tore $\mathbb{T}^2$ et une force $f \in L^2(\mathbb{T}^2)^2$ indépendante du temps. Pour ce travail, on s'appuie notamment sur les travaux de M. Paicu sur les solutions faibles de (3.1) dans V^1. Une des difficultés vient du fait que, dans ce cas, les solutions ne sont a priori pas uniques, et on ne peut donc pas définir les solutions de (3.1) par l'intermédiaire d'un système dynamique classique, tel que celui donné par la définition 1.1, la propriété *(H2)* n'étant pas vérifiée. Pour contourner ce problème, on considère une version affaiblie des systèmes dynamiques, à savoir les semi-groupes généralisés, introduits pas J. Ball (voir par exemple [3] et [4]).

Définition 3.1 *Un semi-groupe généralisé* G *sur un espace métrique* (X, d) *est une famille de fonctions* $\varphi : \mathbb{R}^+ \rightarrow X$ *satisfaisant les hypothèses*

(H1) *Pour tout $z \in X$, il existe au moins une fonction $\varphi \in G$ telle que $\varphi(0) = z$.*

(H2) *Si $\varphi \in G$, alors, pour tout $\tau \geq 0$, $\varphi^\tau \in G$, où $\varphi^\tau(t) = \varphi(t+\tau)$.*

(H3) *Si $\varphi, \psi \in G$ et s'il existe $\tau \geq 0$ tel que $\psi(0) = \varphi(\tau)$, alors la fonction θ définie par*

$$\theta(t) = \begin{cases} \varphi(t), & \textit{pour tout} \quad 0 \leq t \leq \tau, \\ \psi(t-\tau), & \textit{pour tout} \quad t \geq \tau, \end{cases}$$

appartient à G.

(H4) *Si $\varphi_j \in G$ et $\varphi_j(0) \to z$, il existe une sous-suite φ_μ de φ_j et $\varphi \in G$ avec $\varphi(0) = z$ tels que $\varphi_\mu(t) \to \varphi(t)$, pour tout $t \geq 0$.*

Cette définition ne prend notamment pas en compte le caractère unique des solutions que l'on considère, et s'adapte donc bien au cadre des équations des fluides de grade 3 sur V^1_{per}, dans lequel l'unicité des solutions n'est pas connue. Pour un semi-groupe généralisé G, on peut définir une famille d'applications $T(t)$, $t \geq 0$ sur les ensembles de X. Cette famille d'applications est celle de la définition 1.1 pour un système dynamique classique, et est donnée par

$$T(t)x = \{\varphi(t) : \varphi \in G, \varphi(0) = x\},$$

où x est un élément de X, et par

$$T(t)E = \{\varphi(t) : \varphi \in G, \varphi(0) \in E\},$$

où E est un sous ensemble de X.

Dans ce cas, remarquons que $T(t)x$ est un ensemble de X et non plus une seule solution comme c'est le cas pour les systèmes dynamiques classiques. Pour un tel semi-groupe généralisé G et une telle famille d'applications $T(t)$, la notion d'attracteur est la même que celle donnée dans la définition 1.2. Afin de rappeler quelques résultats sur les semi-groupes généralisés, on introduit la définition suivante, qui est classique pour les systèmes dynamiques.

Définition 3.2 *Soit G un semi-groupe généralisé sur un espace métrique X.*

1. *On dit que G est dissipatif point par point s'il existe un ensemble borné B de X tel que, pour tout $\varphi \in G$, il existe $t_0 \geq 0$ tel que $\varphi(t) \in B$, pour tout $t \geq t_0$.*
2. *On dit que G est asymptotiquement compact si, pour toute suite $\varphi_j \in G$ avec $\varphi_j(0)$ borné et toute suite $t_j \in \mathbb{R}^+$ telle que $t_j \to +\infty$, on peut extraire de $\varphi_j(t_j)$ une sous-suite convergente.*

Dans le cadre d'un semi-groupe généralisé G, J. Ball a montré dans [3] le théorème suivant d'existence d'un attracteur suivant, qui est un théorème bien connu dans le cas des systèmes dynamiques classiques.

Théorème 3.5 *Un semi-groupe généralisé G sur un espace métrique X admet un attracteur global si et seulement si G est dissipatif point par point et asymptotiquement compact.*

Ce théorème est l'analogue d'un autre théorème obtenu pour les systèmes dynamiques classiques, et démontré la première fois dans [44]. Dans notre cas, s'il est possible de démontrer que l'ensemble des solutions faibles de (3.1) dans V^1_{per} est un semi-groupe généralisé dissipatif point par point, on ne sait pas à l'heure actuelle si ce semi-groupe est asymptotiquement compact, et ce théorème ne peut donc s'appliquer en l'état. En revanche, si l'on pouvait montrer que la propriété $(H4)$ de la définition 3.1 était vérifiée pour la topologie faible de H^1, alors par une méthode d'égalité d'énergie telle que celle exposée dans [53], on pourrait conclure à la compacité asymptotique. Dans le cadre de cette thèse, nous montrerons un résultat plus faible, qui montre en un sens l'existence d'un attracteur dans V^s_{per}, où $s > 1$, pour la topologie H^1.

Théorème 3.6 *Si $f \in L^2(\mathbb{T}^2)$ est indépendante du temps, $\nu > 0$, $\beta > 0$, $\alpha_1 > 0$ et $\alpha_1 < \sqrt{8\nu\beta}$, alors l'ensemble des solutions faibles de (3.1) sur $\mathbb{T}^2$ à données dans V^1_{per} est un semi-groupe généralisé sur V^1_{per}. Si de plus $\alpha_1 < \sqrt{\nu\beta}$, alors pour tout $s > 1$, il existe un ensemble compact invariant $\mathcal{A}_s \subset V^1_{per}$, qui attire tous les bornés de V^s_{per} pour la topologie H^1.*

Pour montrer que l'ensemble des solutions faibles de (3.1) est un semi-groupe généralisé sur V^1_{per}, on se sert notamment de la méthode d'égalité d'énergie décrite dans [53] combinée à la méthode de monotonie utilisée par M. Paicu pour montrer le théorème 3.2. Pour montrer que les bornés de V^s_{per} sont attirés vers un compact $\mathcal{A}_s$, on montre, en s'appuyant entre autres sur les travaux de V. Busuioc et D. Iftimie, que ces solutions restent bornées dans V^s_{per}, pour peu que $\alpha_1 < \sqrt{\nu\beta}$. En utilisant le caractère compact de V^s_{per} dans V^1_{per} et en reprenant la démonstration du théorème 3.5, on est ensuite à même de conclure à l'existence d'un ensemble compact et invariant de V^1_{per} qui attire les bornés de V^s_{per}. De plus, on verra dans le chapitre 4 que si $1 < s \leq 2$, alors $\mathcal{A}_s$ est un ensemble borné de V^s_{per}.

Ce résultat est plus faible que le théorème 2.6 obtenu pour les fluides de grade 2, car la convergence vers l'attracteur a lieu pour la topologie H^1 alors que les données sont supposées plus régulières que cet espace.

Chapitre 3

Fluides de grade 2

Asymptotic profiles for the second grade fluids equations on $\mathbb{R}^3$

1 Introduction

The equations of fluids of second grade have been introduced from a mathematical point of view in 1974 by J. Dunn and R. Fosdick in [24] and since have been the topic of many research works in mathematics. These fluids are a particular case of a large class of non-Newtonian fluids, called fluids of differential type, or Rivlin-Ericksen fluids (see [59]) which play an important role in the nature. For instance, some oils used in industry or even fluids that we use every day, like wet sand or melted cheese are non-Newtonian fluids. Given the vorticity $\nu > 0$, the parameter $\alpha > 0$ and an initial divergence free vector field u_0 of $\mathbb{R}^2$ or $\mathbb{R}^3$, the equations of motion of fluids of second grade are given by

$$\begin{aligned} &\partial_t (u - \alpha\Delta u) - \nu\Delta u + \text{curl } (u - \alpha\Delta u) \times u + \nabla p = 0, \\ &\text{div } u = 0, \\ &u_{|t=0} = u_0, \end{aligned} \tag{1.1}$$

where $\times$ denotes the classical vectorial product on $\mathbb{R}^3$ and p is the pressure which depends on u. In the two-dimensional case, we have used the convention that $u = (u_1, u_2, 0)$ and $\text{curl } u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$.

Several existence and uniqueness results have been obtained for this system of equations, mainly on a bounded set Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$ with Dirichlet or periodic boundary conditions (see for instance [5], [8], [19], [20], [21], [22], [34], [54] or [52]). The first existence and uniqueness result has been obtained by D. Cioranescu and O. El Hacène in 1984 in [19]. They have shown, on a bounded set of $\mathbb{R}^d$, $d = 2, 3$, with homogeneous boundary conditions, that there exists a unique weak solution to (1.1) belonging to the space $L^\infty\left([0,T], H^3(\Omega)^d\right)$, where $T > 0$ and $H^s(\Omega)$ denotes the Sobolev space of order s (see [19]). Besides, this solution is global in time when the dimension is 2. This result is based on a priori estimates and a Galerkin approximation with a basis of eigenfunctions corresponding to the scalar product associated to the operator curl $(u - \alpha\Delta u)$. In the same case, using the Schauder fixed point theorem, P. Galdi, M. Grobbelaar-Van Dalsen and N. Sauer established the existence and uniqueness of classical solutions to (1.1) when data belong to H^m, with $m \geq 5$ (see [35]). They also have shown that these solutions are global in time, provided that the initial data are small enough in $H^m(\Omega)$. Later, D. Cioranescu and V. Girault improved the results of [19] and [35] and showed that the local weak solutions belonging to $H^3(\Omega)$ are actually global in time in dimension 3 if the data are small enough and are strong solutions if the data belong to H^m, $m \geq 4$ (see [18]). Finally, D. Bresch and J. Lemoine have generalized the results of [35], [19] and [18] in dimension 3 in establishing the existence and uniqueness of local solutions belonging to the space $W^{2,r}(\Omega)$ with $r > 3$. Furthermore, they have shown that these solutions are global in time if the initial data are small enough in $W^{2,r}(\Omega)$ (see [8]). In this work, instead of applying a Galerkin approximation, the authors used Schauder's fixed point Theorem.

In the present paper, we are interested in the description of the asymptotic profiles of the solutions of second grade fluids equations. In what follows, we consider a second grade fluid which fills the whole space $\mathbb{R}^3$, without any forcing term applied to it. In this case, if the initial data are small enough, the solutions of such a system tend to 0 when the time t goes to infinity. The aim of this study is to investigate the way that these solutions go to 0. More precisely, we will show that the solutions of (1.1) behave asymptotically like particular solutions to the heat equation, which are smooth and that one can compute explicitly. In this article, we restrict ourselves to the study of the first order asymptotic profile, that is to say that the speed of the convergence of the solutions of (1.1) to explicit smooth functions is limited by spectral considerations. For Navier-Stokes equations, there exist already several results that describe the asymptotic profiles of the solutions. In dimension 2, T. Gallay and E. Wayne have shown in [36] that the first order asymptotic profiles of the solutions of Navier-Stokes equations are given up to a constant by a smooth Gaussian function that is called the Oseen Vortex sheet.

Actually, the equations that they considered are the scalar vorticity equations, and not Navier-Stokes equations themselves. This result initially held under restrictions on the size of the data, but has been generalized to the case of any data in [38]. For this work, the authors applied arguments that come from the study of dynamical systems. In fact, they have shown the existence of a finite-dimensional manifold locally invariant by the semiflow associated to the Navier-Stokes equations. Then, they proved that the solutions of Navier-Stokes equations are locally attracted by this manifold, and consequently behave like the solutions on it. The study of the dynamics of Navier-Stokes equation onto this manifold gave them the description of the first and second order asymptotic profiles. In dimension 3, with similar methods, they have shown the same kind of results in [37], under smallness assumptions on the size of the data. The asymptotic profiles of the solutions of the equations of second grade fluids have been studied in dimension 2 by B. Jaffal-Mourtada in [47]. She has shown, under smallness assumptions on the data, that the first order asymptotic profiles of solutions to second grade fluids equations are given up to a constant by the Oseen vortex sheet, as it is the case for Navier-Stokes equations in $\mathbb{R}^2$. To obtain this result, the author performed energy estimates in various functions spaces, notably weighted Sobolev spaces. The concrete interpretation of this result is that, in dimension 2, the fluids of second grade behave asymptotically like Newtonian fluids. The main aim of this article is to extend this observation to the dimension 3.

Actually, the system that we will solve in this article is not exactly (1.1) but the one satisfied by the vorticity $w = \operatorname{curl} u$. Assuming, for the sake of simplicity, that $\nu = 1$, considering initial vorticity data w_0 and taking formally the curl of (1.1), we get the vorticity system of equations

$$\begin{aligned} &\partial_t (w - \alpha \Delta w) - \Delta w + \operatorname{curl}\ ((w - \alpha \Delta w) \times u) = 0, \\ &\operatorname{div} u = \operatorname{div} w = 0, \\ &w_{|t=0} = w_0. \end{aligned} \tag{1.2}$$

In this system, the divergence free vector field u is reconstructed from w via the Biot-Savart law, which is a way to get a divergence free vector field from its given vorticity. In the section 2, more details will be given about the Biot-Savart law and its properties.

In this article, we show that the solutions of the system (1.2) behave asymptotically like vector fields whose components are self-similar solutions to the well known heat equations, that is to say under the form

$$(t,x) \to \frac{1}{(t+T)^2} F\left(\frac{x}{\sqrt{t+T}}\right),$$

where F is a divergence free vector field of $\mathbb{R}^3$ and T is a positive constant.

We introduce now a powerful tool in the studying of asymptotics of solutions to partial differential equations, that is scaled variables or self-similar variables. Let T be a positive constant that we will always assume $T \geq 1$ and w be a solution of (1.2). We make the change of variable $X = \dfrac{x}{\sqrt{t+T}}$ and set $\tau = \log(t+T)$. By this way, one defines W and U, given by

$$\begin{aligned} w\left(t,x\right) &= \frac{1}{t+T} W\left(\log\left(t+T\right), \frac{x}{\sqrt{t+T}}\right), \\ u\left(t,x\right) &= \frac{1}{\sqrt{t+T}} U\left(\log\left(t+T\right), \frac{x}{\sqrt{t+T}}\right). \end{aligned} \tag{1.3}$$

Equivalently, we have the equalities

$$\begin{aligned} W\left(\tau,X\right) &= e^{\tau} w\left(e^{\tau}-T, e^{\tau/2}X\right), \\ U\left(\tau,X\right) &= e^{\tau/2} u\left(e^{\tau}-T, e^{\tau/2}X\right). \end{aligned} \tag{1.4}$$

Scaling variables have been initially introduced to study the asymptotic behaviours of solutions of parabolic equations, and in particular to show the convergence to self-similar solutions (see [25], [26], [33] or [48]). Actually, this tool is also efficient to study the long-time behaviour of a lot of various equations, not necessarily parabolic ones. For instance, T. Gallay and G. Raugel used them to describe the first and second order asymptotic profiles of solutions to damped waved equations (see [40]) and to show the stability of hyperbolic fronts (see [41]). Self-similar variables have been also used to investigate the asymptotic profiles of the solutions of Navier-Stokes equations in [36], [37], [38] and [39] and second grade fluids equations in dimension 2 in [47]. Assuming that w is a solution of (1.2), a short computation shows that W is a solution of the system

$$\begin{aligned} &\partial_\tau \left(W - \alpha e^{-\tau}\Delta W\right) - \mathcal{L}(W) + \text{curl }\left(\left(W - \alpha e^{-\tau}\Delta W\right) \times U\right) \\ &\qquad\qquad\qquad\qquad + \alpha e^{-\tau}\Delta W + \alpha e^{-\tau}\tfrac{X}{2}.\nabla\Delta W = 0, \\ &\text{div } U = \text{div } W = 0, \\ &W_{|\tau=\log(T)} = W_0, \end{aligned} \tag{1.5}$$

where $\mathcal{L}$ is the linear differential operator defined by

$$\mathcal{L}(W) = \Delta W + W + \tfrac{X}{2}.\nabla W.$$

In the first equality of the previous system, there are several terms which formally tend to 0 when time goes to infinity. Actually, the main theorem of this article shows that the solutions of (1.5) converge when τ goes to infinity to particular solutions to the equality

$$\partial_\tau W_\infty = \mathcal{L}(W_\infty). \tag{1.6}$$

More precisely, the aim of the present paper is to decompose W on the spectrum of $\mathcal{L}$ on an appropriate space of functions and to show that the asymptotic behaviour of W is dominated by the projection of W onto the eigenspace corresponding to the first eigenvalue of $\mathcal{L}$. Additionally, this projection satisfies the equality (1.6). We define now the weighted Lebesgue spaces, which are suitable for the study of the spectrum of $\mathcal{L}$. For every $m \in \mathbb{N}$, one defines $L^2(m)$, given by

$$L^2(m) = \left\{ u \in L^2(\mathbb{R}^3) : \left(1 + |x|^2\right)^{m/2} u \in L^2(\mathbb{R}^3) \right\},$$

where $|x| = \left(\displaystyle\sum_{i=1}^{3} x_i^2 \right)^{1/2}$.

By the same way, for $m \in \mathbb{N}$ and $n \geq 2$, we define the weighted Sobolev spaces by

$$\begin{aligned} H^1(m) &= \left\{ u \in L^2(m) : \partial_i u \in L^2(m), i \in \{1,2,3\} \right\}, \\ H^n(m) &= \left\{ u \in L^2(m) : \partial_i u \in H^{n-1}(m), i \in \{1,2,3\} \right\}. \end{aligned}$$

The incompressibility condition on the vector fields W and U makes natural to work on the spaces

$$\mathbb{L}^2(m) = \left\{ u \in L^2(m)^3 : \operatorname{div} u = 0 \right\},$$

$$\mathbb{H}^2(m) = \left\{ u \in H^2(m)^3 : \operatorname{div} u = 0 \right\},$$

equipped with the norms

$$\|u\|_{L^2(m)} = \left\| \left(1 + |x|^2\right)^{\frac{m}{2}} u \right\|_{L^2},$$

and

$$\|u\|_{H^2(m)} = \left(\|u\|_{L^2(m)}^2 + \|\nabla u\|_{L^2(m)}^2 + \|\nabla^2 u\|_{L^2(m)}^2 \right)^{1/2}.$$

In [37], T. Gallay and E. Wayne show that the spectrum of $\mathcal{L}$ on $\mathbb{L}^2(m)$ is the union of the discrete spectrum

$$\sigma_d(\mathcal{L}) = \left\{ -\tfrac{1}{2}(k+1), k \in \mathbb{N}^* \right\},$$

and the continuous one

$$\sigma_c(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : Re(\lambda) \leq \tfrac{1}{4} - \tfrac{m}{2}\right\}.$$

In order to describe the first order asymptotic profile of solutions of (1.5), we need to have at least one isolated eigenvalue. Looking at $\sigma_c(\mathcal{L})$, we notice that one can "push" the continuous spectrum to the left by choosing m large enough. For this reason, we should work at least in the weighted space $\mathbb{L}^2(3)$, where -1 is an isolated eigenvalue of $\mathcal{L}$. Actually, in order to be close to the optimal rate of convergence, we prefer working in $\mathbb{L}^2(4)$, where the discrete spectrum is $\sigma_d(\mathcal{L}) = \left\{-1, -\frac{3}{2}\right\}$ and the continuous one is $\sigma_c(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : Re(\lambda) \leq -\frac{7}{4}\right\}$. The main aim of this article is to show that one can decompose a solution W of (1.5) into the form

$$W(\tau) = \Omega(\tau) + R(\tau), \tag{1.7}$$

where Ω is an eigenfunction of $\mathcal{L}$ associated to the eigenvalue -1 and R tends to 0 faster than Ω into $\mathbb{L}^2(4)$ when τ goes to infinity.

Since the first eigenvalue smaller than -1 is $-\frac{3}{2}$, the best result that one expects is

$$R(\tau) = \mathcal{O}(e^{-\frac{3\tau}{2}}) \text{ in } \mathbb{L}^2(4).$$

Actually, the result that we obtain holds under smallness assumptions on the size of the data in $\mathbb{H}^2(4)$. Besides, provided that the initial data are small enough compared to the parameters of the equations, one can be as close as wanted to the optimal rate.

2 First order asymptotics and preliminary results

Before stating the main theorem of this paper, we have describe the eigenspace of $\mathcal{L}$ associated to the eigenvalue -1. In [37, appendix A], they show that the multiplicity of the eigenvalue -1 is 3 and that a suitable basis $\{f_1, f_2, f_3\}$ of the associated eigenspace E_{-1} is given by

$$f_i = \text{curl}\,(Ge_i), \quad i = 1, 2, 3, \tag{2.1}$$

where $G(X) = \dfrac{1}{(4\pi)^{3/2}} e^{-\frac{|X|^2}{4}}$ and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

Through a short computation, we see that $f_i(X) = p_i(X)G(X), \quad i = 1, 2, 3,$ where

$$p_1(X) = \frac{1}{2}\begin{pmatrix} 0 \\ -X_3 \\ X_2 \end{pmatrix}, \; p_2(X) = \frac{1}{2}\begin{pmatrix} X_3 \\ 0 \\ -X_1 \end{pmatrix} \text{ and } p_3(X) = \frac{1}{2}\begin{pmatrix} -X_2 \\ X_1 \\ 0 \end{pmatrix}.$$

In particular, the vector fields p_i satisfy $\quad \operatorname{div} p_i = 0 \quad$ and $\quad \operatorname{curl} p_i = e_i$. Integrating by parts, we notice also that

$$\int_{\mathbb{R}^3} p_i(X).f_j(X)dX = \int_{\mathbb{R}^3} \operatorname{curl}\left(p_i(X)\right).\left(G(X)e_j\right)dX = (e_i.e_j)\int_{\mathbb{R}^3} G(X)dX = \delta_{ij}. \quad (2.2)$$

Furthermore, defining $\mathcal{L}^* = \Delta - \frac{X}{2}.\nabla - \frac{1}{2}$ the formal adjoint of $\mathcal{L}$, we check easily that

$$\mathcal{L}^* p_i = -p_i.$$

With the basis $\{f_1, f_2, f_3\}$, the decomposition (1.7) can be written

$$W(\tau) = \sum_{i=1}^{3} \beta_i(\tau) f_i + R(\tau), \tag{2.3}$$

where $\beta_i(\tau) \in \mathbb{R}$.

As we can see in [37], $\mathcal{L}^2(4) = E_{-1} \oplus \mathcal{W}$, where

$$\mathcal{W} = \left\{ f \in \mathcal{L}^2(4) : \int_{\mathbb{R}^3} X_i f_j(X)dX = 0, \quad i,j = 1,2,3 \right\}.$$

Consequently, one has to choose β_i such that $\displaystyle\int_{\mathbb{R}^3} X_i R_j(\tau, X)dX = 0$, for $i,j \in \{1,2,3\}$. To this end, we set

$$\beta_i(\tau) = \int_{\mathbb{R}^3} p_i(X).W(\tau, X)dX.$$

In fact, assuming that $W \in \mathbb{L}^2(4)$ and using the divergence free property of W, it is easy to check that

$$\int_{\mathbb{R}^3} p_1(X).W(X)dX = \int_{\mathbb{R}^3} X_2 W_3(X)dX = -\int_{\mathbb{R}^3} X_3 W_2(X)dX,$$
$$\int_{\mathbb{R}^3} p_2(X).W(X)dX = \int_{\mathbb{R}^3} X_3 W_1(X)dX = -\int_{\mathbb{R}^3} X_1 W_3(X)dX,$$
$$\int_{\mathbb{R}^3} p_3(X).W(X)dX = \int_{\mathbb{R}^3} X_1 W_2(X)dX = -\int_{\mathbb{R}^3} X_2 W_1(X)dX,$$

and thus, using (2.2) and the decomposition (2.3), we can conclude that

$$\int_{\mathbb{R}^3} X_i R_j(X)dX = 0, \text{ for all } i,j \in \{1,2,3\}.$$

The next lemma gives more details about β_i, and shows that the projection of W onto E_{-1} is actually a solution of (1.6).

Lemma 2.1 *Let $W \in C^o\left([\tau_0, T), \mathbb{H}^2(4)\right)$ be a solution of (1.5) and let*

$$\beta_i(\tau) = \int_{\mathbb{R}^3} p_i(X).W(\tau, X)dX.$$

Then, for all $\tau \in [\tau_0, T]$,

$$\beta_i(\tau) = b_i e^{-\tau}, \tag{2.4}$$

where $b_i = \displaystyle\int_{\mathbb{R}^3} p_i(X).W_0(X)dX$.

Proof: The proof of this lemma is made formally, assuming that every quantity that we consider is well defined. Actually, in the remaining of this article, we will work with regularized solutions for which the next computations are rigorous. In order to get (2.4), we only have to show that β_i satisfies

$$\partial_\tau \beta_i\left(\tau\right) = -\beta_i(\tau). \tag{2.5}$$

Performing the L^2-scalar product of the first equality of (1.5) with p_i, we obtain

$$\begin{aligned}\partial_\tau\beta_i(\tau) = {} & \alpha e^{-\tau}\left(p_i, \partial_\tau \Delta W\right)_{L^2} - \alpha e^{-\tau}\left(p_i, \Delta W\right)_{L^2} + \left(p_i, \mathcal{L}(W)\right)_{L^2} \\ & + \left(p_i, \operatorname{curl}\ \left(U \times \left(W - \alpha e^{-\tau}\Delta W\right)\right)\right)_{L^2} - \alpha e^{-\tau}\left(p_i, \Delta W + \tfrac{X}{2}.\nabla\Delta W\right)_{L^2}.\end{aligned} \tag{2.6}$$

Integrating several times by parts, it is easy to check that

$$\alpha e^{-\tau}\left(p_i, \partial_\tau\Delta W\right)_{L^2} = \alpha e^{-\tau}\left(p_i, \Delta W\right)_{L^2} = \alpha e^{-\tau}\left(p_i, \Delta W + \tfrac{X}{2}.\nabla\Delta W\right)_{L^2} = 0.$$

Thus we have

$$\partial_\tau\beta_i(\tau) = -\beta_i(\tau) + \int_{\mathbb{R}^3} e_i.\left(U(X) \times \left(W(X) - \alpha e^{-\tau}\Delta W(X)\right)\right)dX. \tag{2.7}$$

It remains to show that the last term of the right hand size of (2.7) vanishes. Noticing that $W = \operatorname{curl} U$, an easy computation shows, for $i \in \{1,2,3\}$,

$$\left(U(X) \times \left(W(X) - \alpha e^{-\tau}\Delta W(X)\right)\right)_i = \frac{1}{2}\partial_i\left(|U|^2\right) - U.\nabla U_i - \alpha e^{-\tau}\left(U.\partial_i\Delta U - U.\nabla\Delta U_i\right). \tag{2.8}$$

Thus, using the divergence free property of U and integrating by parts, we get

$$\int_{\mathbb{R}^3} e_i.\left(U(X) \times \left(W(X) - \alpha e^{-\tau}\Delta W(X)\right)\right) dX = -\alpha e^{-\tau} \int_{\mathbb{R}^3} U(X).\partial_i \Delta U(X) dX.$$

Another integration by parts yields

$$\int_{\mathbb{R}^3} e_i.\left(U(X) \times \left(W(X) - \alpha e^{-\tau}\Delta W(X)\right)\right) dX = \frac{\alpha}{2} e^{-\tau} \int_{\mathbb{R}^3} \partial_i \left(|\nabla U(X)|^2\right) dX = 0,$$

and thus we obtain (2.5).

□

We are now able to state the main theorem of this paper, which shows in particular that the first order asymptotic profile of a solution W in $\mathbb{H}^2(4)$ of (1.5) is the same as the first order asymptotic profile obtained for Navier-Stokes equations.

Theorem 2.1 *Let θ be a fixed constant such that $0 < \theta < \frac{3}{2}$ and $W_0 \in \mathbb{H}^2(4)$. There exist two positive constants $\gamma_0 = \gamma_0(\alpha)$ and $T_0 = T_0(\alpha) \geq 1$ such that if $T \geq T_0$ and there exists a positive constant $\gamma \leq \gamma_0$ such that*

$$\|W_0\|^2_{L^2(4)} + \|\nabla W_0\|^2_{L^2} + \alpha e^{-\tau_0} \|\Delta W_0\|^2_{L^2} + \alpha^2 e^{-2\tau_0} \left\||X|^4 \Delta W_0\right\|^2_{L^2} \leq \gamma \left(\frac{3}{2} - \theta\right)^2, \quad (2.9)$$

where $\tau_0 = \log(T)$,

then there exist a unique solution $W \in C^0\left([\tau_0, +\infty), \mathbb{H}^2(4)\right)$ to the system (1.5) and a positive constant $C = C(\theta, \alpha, \gamma, T_0)$ such that

$$\left\|\left(Id - \alpha e^{-\tau}\Delta\right)\left(W(\tau) - e^{-\tau}\sum_{i=1}^{3} b_i f_i\right)\right\|_{L^2(4)} \leq C e^{-\theta\tau}, \quad (2.10)$$

where $b_i = \displaystyle\int_{\mathbb{R}^3} p_i(X).W_0(X) dX$.

In the classical variables, the theorem 2.1 gives the next corollary.

Corollary 2.1 *Let θ be a constant such that $0 < \theta < \frac{3}{2}$, $w_0 \in \mathbb{H}^2(4)$ and $b_i = \dfrac{1}{T}\displaystyle\int_{\mathbb{R}^3} p_i(x).w_0(x) dx$. There exists $\gamma_0 = \gamma_0(\alpha) > 0$ and $T_0 = T_0(\alpha) \geq 1$ such that if there exist $T \geq T_0$ and $\gamma \leq \gamma_0$ such that*

$$\begin{aligned} T^{1/2} \|w_0\|^2_{L^2} + T^{-7/2} \left\||x|^4 w_0\right\|^2_{L^2} + T^{3/2} \|\nabla w_0\|^2_{L^2} \\ + \alpha T^{3/2} \|\Delta w_0\|^2_{L^2} + \alpha^2 T^{-3/2} \left\||x|^4 \Delta w_0\right\|^2_{L^2} \leq \gamma \left(\tfrac{3}{2} - \theta\right)^2, \end{aligned} \quad (2.11)$$

then there exists a unique solution $w \in C^0([0,+\infty), \mathbb{H}^2(4))$ *to the system (1.2) such that, for all* $1 \leq p \leq 2$, *there exists a positive constant* $C = C(\theta, \alpha, \gamma, T_0, p)$ *such that*

$$\left\| w(t) - \sum_{i=1}^{3} \frac{b_i}{(t+T)^2} f_i \left(\frac{x}{\sqrt{t+T}} \right) \right\|_{L^p} \leq C\,(t+T)^{-1-\theta+\frac{3}{2p}}, \tag{2.12}$$

and there exists a positive constant $C = C(\theta, \alpha, \gamma, T_0)$ *such that*

$$\left\| |x|^4 \left(Id - \alpha\Delta\right) \left(w(t) - \sum_{i=1}^{3} \frac{b_i}{(t+T)^2} f_i \left(\frac{x}{\sqrt{t+T}} \right) \right) \right\|_{L^2} \leq C\,(t+T)^{\frac{1}{4}-\theta}. \tag{2.13}$$

Let u *be the divergence free vector field obtained from* w *through the Biots-Savart law. For all* $p \in \left[\frac{3}{2}, 6\right]$, *there exists a positive constant* $C = C(\theta, \alpha, \gamma, T_0, p)$ *such that*

$$\left\| u(t) - \sum_{i=1}^{3} \frac{b_i}{(t+T)^{3/2}} v_i \left(\frac{x}{\sqrt{t+T}} \right) \right\|_{L^p} \leq C\,(t+T)^{-\frac{1}{2}-\theta+\frac{1}{p}}, \tag{2.14}$$

where v_i *is obtained from* f_i *via the Biots-Savart law.*

We prove the theorem 2.1 in several steps. First, in section 3, we introduce a new system that is close to (1.5), but which contains the regularizing term $\varepsilon\Delta^2 W$, with ε a small positive constant that is devoted to tend to 0. Thanks to this regularizing term, we are able, through a semi-group method, to show the existence of local solutions to the regularized system. In a second time, in the section 4 we make energy estimates on these approximate solutions, and show that these ones are global in time and satisfy the inequality (2.10). Then, in section 5, we pass to the limit when ε tends to 0 and show that the approximate solutions converge to a global solution of (1.5) which satisfies (2.10). Finally, we show that this solution is unique.

Biot-Savart law:

Now, we recall some properties of the Biot-Savart law. Let w be a given divergence free vector field of $\mathbb{R}^3$, the Biot-Savart law gives a divergence free vector field u such that $\operatorname{curl} u = w$. It is given by

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times w(y)}{|x-y|^3} dy. \tag{2.15}$$

In particular, passing into scaled variables preserves the Biot-Savart law. Indeed, if u is obtained from w via the Biot-Savart law and W is w expressed into scaled variables, then the divergence free vector field U obtained from W through the Biot-Savart law is u expressed in scaled variables. The next lemma gives some estimates on vector fields obtained by the Biot-Savart law, in various functions spaces.

Lemma 2.2 *Let u be the velocity field obtained from w via the Biot-Savart law (2.15).*
(a) Assume that $1 < p < 3$, $\frac{3}{2} < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. If $w \in L^p(\mathbb{R}^3)^3$, then $u \in L^q(\mathbb{R}^3)^3$, and there exists $C > 0$ such that

$$\|u\|_{L^q} \leq C \|w\|_{L^p} . \tag{2.16}$$

(b) Assume that $1 \leq p < 3 < q \leq \infty$, and define $\eta \in (0,1)$ by the relation $\frac{1}{3} = \frac{\eta}{p} + \frac{(1-\eta)}{q}$. If $w \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$, then $u \in L^\infty(\mathbb{R}^3)^3$ and there exists $C > 0$ such that

$$\|u\|_{L^\infty} \leq C \|w\|_{L^p}^{\eta} \|w\|_{L^q}^{1-\eta} . \tag{2.17}$$

(c) Assume that $1 < p < \infty$. If $w \in L^p(\mathbb{R}^3)^3$, then $\nabla u \in L^p(\mathbb{R}^3)^9$ and there exists $C > 0$ such that

$$\|\nabla u\|_{L^p} \leq C \|w\|_{L^p} . \tag{2.18}$$

This lemma is proved in [37] and will be very useful when making estimates on solutions of (1.5).

3 Approximate solutions

In this section, we introduce a new system that is close to (1.2), which contains the regularizing term $\varepsilon\Delta^2 w$, where ε is a small positive constant. The reason to introduce such a system is to get smooth solutions of the new system, for which we are able to make estimates in $\mathbb{H}^2(4)$ and obtain the inequality (2.10). In the section 5, we will pass to the limit when ε goes to 0 and show that the limit of the solution of the regularized system satisfies also the inequality (2.10). We introduce the following regularized system, given by

$$\begin{aligned} &\partial_t (w_\varepsilon - \alpha\Delta w_\varepsilon) + \varepsilon\Delta^2 w_\varepsilon - \Delta w_\varepsilon + \text{curl } ((w_\varepsilon - \alpha\Delta w_\varepsilon) \times u_\varepsilon) = 0, \\ &\text{div } u_\varepsilon = \text{div } w_\varepsilon = 0, \\ &w_{\varepsilon|t=0} = w_0. \end{aligned} \tag{3.1}$$

The next theorem shows that, for every $w_0 \in \mathbb{H}^2(4)$, there exists a unique local solution to (3.1) belonging to $\mathbb{H}^2(4)$, which is smooth enough to perform the estimates of the section 4.

Theorem 3.1 *Let $\varepsilon > 0$ and $w_0 \in \mathbb{H}^2(4)$. There exists $t_\varepsilon > 0$ and a unique solution w_ε to the system (3.1) defined on the time interval $[0, t_\varepsilon)$ such that*

$$w_\varepsilon \in C^1\left((0,t_\varepsilon), \mathbb{H}^1(4)\right) \cap C^0\left((0,t_\varepsilon), \mathbb{H}^3(4)\right).$$

Proof: To get this result, one defines $w_{\varepsilon,\mu}(t,x) = w_\varepsilon\left(t, \frac{x}{\mu}\right)$, where $\mu > 0$. This change of variables enables us to show the existence of solutions to the system (3.1) without restrictions on the size of the parameter α. We define $u_{\varepsilon,\mu}$ obtained from $w_{\varepsilon,\mu}$ by the Biot-Savart law (2.15). It is easy to check that $u_{\varepsilon,\mu}(t,x) = \mu u_\varepsilon(t, \frac{x}{\mu})$. In order to show the existence of a unique solution to (3.1), we will prove that there exists a unique solution to the system

$$\begin{array}{l} \partial_t\left(w_{\varepsilon,\mu} - \alpha\mu^2\Delta w_{\varepsilon,\mu}\right) - \varepsilon\mu^4\Delta^2 w_{\varepsilon,\mu} - \mu^2\Delta w_{\varepsilon,\mu} + \text{curl}\ \left((w_{\varepsilon,\mu} - \alpha\mu^2\Delta w_{\varepsilon,\mu}) \times u_{\varepsilon,\mu}\right) = 0, \\ \text{div}\ w_{\varepsilon,\mu} = \text{div}\ u_{\varepsilon,\mu} = 0, \\ w_{\varepsilon,\mu|t=0} = w_0(\frac{x}{\mu}) \in \mathbb{H}^2(4). \end{array} \tag{3.2}$$

We define now $z_\varepsilon(t,x) = q(x) w_{\varepsilon,\mu}(t,x)$, where $q(x) = \left(1 + |x|^4\right)$. In particular, if $w_{\varepsilon,\mu} \in \mathbb{L}^2(4)$, then $z_\varepsilon \in \mathcal{H}$, where

$$\mathcal{H} = \left\{z \in L^2(\mathbb{R}^3)^3 : \text{div}\ (q^{-1}z) = 0\right\}.$$

For later use, we define also, for $s \geq 0$,

$$\mathcal{H}^s = \mathcal{H} \cap H^s(\mathbb{R}^3)^3, \quad \text{and} \quad \mathcal{H}^{-s} = \left(\mathcal{H}^s\right)',$$

where $\left(\mathcal{H}^s\right)'$ denotes the dual space of $\mathcal{H}^s$.

We equip $\mathcal{H}^s$ with the classical H^s Sobolev norm, which makes $\mathcal{H}^s$ complete. From the system (3.2), we deduce the following one, that we solve in z_ε,

$$\begin{array}{l} \partial_\tau\left(z_\varepsilon - \alpha\mu^2\Delta z_\varepsilon - \alpha\mu^2 q\Delta q^{-1} z_\varepsilon - 2\alpha\mu^2 q\nabla q^{-1}.\nabla z_\varepsilon\right) + \varepsilon\mu^4\Delta^2 z_\varepsilon = F\left(x, z_\varepsilon\right), \\ \text{div}\ (q^{-1}z_\varepsilon) = 0, \\ z_{\varepsilon|t=0}(x) = z_0(x) \in \mathcal{H}^2, \end{array} \tag{3.3}$$

where

$$\begin{array}{rl} F(x,z_\varepsilon) = & -\varepsilon\mu^4 q\Delta^2\left(q^{-1}z_\varepsilon\right) + \mu^2 q\Delta\left(q^{-1}z_\varepsilon\right) \\ & +q\text{curl}\ \left(u_{\varepsilon,\mu} \times \left(q^{-1}z_\varepsilon - \mu^2\alpha_1\Delta\left(q^{-1}z_\varepsilon\right)\right)\right). \end{array}$$

The system (3.3) is actually autonomous. Indeed, one can recover $u_{\varepsilon,\mu}$ by the Biot-Savart law (2.15) applied to $q^{-1}z_\varepsilon$. To show the existence of solutions to (3.1) in $\mathbb{H}^1(4)$, it suffices to show the existence of solutions to (3.3) in $\mathcal{H}^1$, for data belonging to $\mathcal{H}^2$.

We set two linear differential operators $B : D(B) = \mathcal{H}^1 \to \mathcal{H}^{-1}$ and $D : D(D) = \mathcal{H} \to \mathcal{H}^{-1}$, given by

$$\begin{aligned}B &= \alpha\mu^2 q\Delta q^{-1} + \alpha\mu^2\Delta,\\ D &= \alpha\mu^2 q\nabla q^{-1}.\nabla.\end{aligned}$$

Via Lax-Milgram theorem, we show now that if μ is sufficiently small with respect to α, the operator $(I - B - D)$ is invertible. In order to do that, we define the bilinear form on $\mathcal{H}^1 \times \mathcal{H}^1$, given by

$$a(u,v) = (u,v)_{L^2} + \alpha\mu^2 (\nabla u, \nabla v)_{L^2} - \alpha\mu^2 (q\Delta q^{-1}u, v)_{L^2} - 2\alpha\mu^2 (q\nabla q^{-1}.\nabla u, v)_{L^2}.$$

Since $q\Delta q^{-1}$ and $q\nabla q^{-1}$ are bounded on $\mathbb{R}^3$, the bilinear form a is continuous on $\mathcal{H}^1$. We now show, taking μ small enough, that a is also coercive on $\mathcal{H}^1$. Indeed, integrating by parts and using Hölder and Young inequalities, we have

$$a(u,u) \geq \left(1 - \alpha\mu^2 \sup_{x\in\mathbb{R}^3} \left(q\Delta q^{-1}\right) + \alpha\mu^2 \inf_{x\in\mathbb{R}^3} \left(\operatorname{div}\left(q\nabla q^{-1}\right)\right)\right) \|u\|_{L^2}^2 + \alpha\mu^2 \|\nabla u\|_{L^2}^2.$$

Thus, if we take μ sufficiently small, we get

$$a(u,u) \geq C(\alpha,\mu) \|u\|_{H^1}^2,$$

where $C(\alpha,\mu)$ is a positive constant depending on α and μ.

The classical Lax-Milgram theorem enables us to define $(I - B - D)^{-1}$ from $\mathcal{H}^{-1}$ to $\mathcal{H}^1$. We define the linear differential operator $A : D(A) = \mathcal{H}^3 \to \mathcal{H}^1$ given by

$$A = \varepsilon\mu^4 (I - B - D)^{-1} \Delta^2.$$

We can rewrite the system (3.3) as follows:

$$\begin{aligned}\partial_\tau z_\varepsilon + Az_\varepsilon &= (I - B - D)^{-1} F(x, z_\varepsilon),\\ z_{\varepsilon|t=0} &= z_0.\end{aligned} \tag{3.4}$$

In order to show the existence of solutions to such a system, we use, like in [47], a semi-group method. First, we show that $-A$ generates an analytic semi-group on $\mathcal{H}^1$ which is equivalent as A is sectorial on $\mathcal{H}^1$. We decompose A as follows:

$$\begin{aligned}A &= \varepsilon\mu^4 \left(Id - B - D\right)^{-1} \Delta^2 \\ &= \varepsilon\mu^4 \left(Id - B\right)^{-1} \Delta^2 + \varepsilon\mu^4 \left(Id - B - D\right)^{-1} D \left(Id - B\right)^{-1} \Delta^2 \\ &= J + R,\end{aligned}$$

where

$$\begin{aligned}J &= Id + \varepsilon\mu^4 \left(Id - B\right)^{-1} \Delta^2, \\ R &= -Id + \varepsilon\mu^4 \left(Id - B - D\right)^{-1} D \left(Id - B\right)^{-1} \Delta^2.\end{aligned}$$

We first show that J is sectorial. We will see later that R satisfies properties that enable to conclude that A is sectorial if J is sectorial. Taking μ sufficiently small compared to α, it is easy, arguing like we did to invert $(I - B - D)$, to show that $(I - B)^{-1}$ is well defined from $\mathcal{H}^{-1}$ to $\mathcal{H}^1$. Consequently, the operator J is well defined from $\mathcal{H}^3$ to $\mathcal{H}^1$. We define now the bilinear form j on $\mathcal{H}^2 \times \mathcal{H}^2$ associated to J. To this end, we introduce a H^1-scalar product which is adapted to J. We define

$$\langle u, v\rangle_{H^1} = \left(\left(1 - \alpha\mu^2 q\Delta q^{-1}\right) u, v\right)_{L^2} + \alpha\mu^2 \left(\nabla u, \nabla v\right)_{L^2}.$$

If μ is sufficiently small, $\langle ., .\rangle_{H^1}$ is a scalar product on $\mathcal{H}^1$. In particular, if $u \in \mathcal{H}^2$ and $v \in \mathcal{H}^1$, one has

$$\langle u, v\rangle_{H^1} = \left(\left(I - B\right) u, v\right)_{L^2}.$$

Via this product, we define

$$j(u, v) = \langle u, v\rangle_{H^1} + \varepsilon\mu^4 \left(\Delta u, \Delta v\right).$$

In particular, if $u \in \mathcal{H}^3$ and $v \in \mathcal{H}^1$, one has

$$j(u, v) = \langle Ju, v\rangle_{H^1}.$$

The bilinear form j is obviously continuous on $\mathcal{H}^2 \times \mathcal{H}^2$. Furthermore, if μ is small enough, it is also coercive on $\mathcal{H}^2$. Indeed,

$$\begin{aligned}j(u, u) &\geq C(\alpha, \mu) \left\|u\right\|_{H^1}^2 + \varepsilon\mu^4 \left\|\Delta u\right\|_{L^2}^2 \\ &\geq C(\alpha, \mu, \varepsilon) \left\|u\right\|_{H^2}^2.\end{aligned}$$

Thus j is continuous and coercive on $\mathcal{H}^2$ and consequently J is sectorial on $\mathcal{H}^1$, that is equivalent to say that $-J$ generates an analytic semi-group on $\mathcal{H}^1$. Furthermore, we can check that R is continuous from $\mathcal{H}^2$ to $\mathcal{H}^1$, and we have

$$\|Ru\|_{H^1} \leq C(\alpha,\mu,\varepsilon)\,\|u\|_{H^2}\,.$$

Using the coerciveness of j, we get, for all $u \in \mathcal{H}^3$,

$$\begin{aligned} \|Ru\|_{H^1}^2 &\leq C(\alpha,\mu,\varepsilon) j(u,u) \\ &\leq C(\alpha,\mu,\varepsilon)\,\langle Ju,u\rangle_{H^1} \\ &\leq C\,\|Ju\|_{H^1}\,\|u\|_{H^1}\,. \end{aligned} \tag{3.5}$$

Applying the Young inequality, we obtain, for all $\delta > 0$

$$\|Ru\|_{H^1}^2 \leq \delta\,\|Ju\|_{H^1}^2 + C\,\|u\|_{H^1}^2\,, \text{ for all } u \in \mathcal{H}^3.$$

From a classical result that we can find in the book of D. Henry [46], it implies that $J+R$ is sectorial on $\mathcal{H}^1$.

To achieve this proof, we check that $A^{-1}F(x,v)$ is locally Lipschitz in $v \in \mathcal{H}^1$ on the bounded sets of $\mathcal{H}^2$. According to [58, section 6.3] and [46, chapter 3], we finally get Theorem 3.1.

$\square$

4 Energy estimates

In this section, we perform energy estimates on the solution of (3.1) given by Theorem 3.1. We consider a fixed positive constant θ such that $0 < \theta < \frac{3}{2}$, which is the rate of convergence of Theorem 2.1. Let T be a positive constant which will be made more precise later and that we assume, without loss of generality, to be such that $T \geq 1$. We consider W_ε the divergence free vector field obtained from w_ε via the change of variables (1.4). According to Theorem 5.1, there exists a maximal time τ_ε such that W_ε belongs to $C^1\left([\tau_0,\tau_\varepsilon)\,,\mathbb{H}^1(4)\right) \cap C^0\left([\tau_0,\tau_\varepsilon)\,,\mathbb{H}^3(4)\right)$, where $\tau_0 = \ln(T)$. A short computation shows that W_ε is the solution of the system

$$\begin{aligned} &\partial_\tau\left(W_\varepsilon - \alpha e^{-\tau}\Delta W_\varepsilon\right) + \varepsilon e^{-\tau}\Delta^2 W_\varepsilon - \mathcal{L}(W_\varepsilon) + \mathrm{curl}\,\left(\left(W_\varepsilon - \alpha e^{-\tau}\Delta W_\varepsilon\right)\times U_\varepsilon\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \alpha e^{-\tau}\Delta W_\varepsilon + \alpha e^{-\tau}\tfrac{X}{2}.\nabla\Delta W_\varepsilon = 0, \\ &\mathrm{div}\,U_\varepsilon = \mathrm{div}\,W_\varepsilon = 0, \\ &W_{\varepsilon|\tau=\tau_0} = W_0, \end{aligned} \tag{4.1}$$

where we recall that

$$\mathcal{L}(W_\varepsilon) = W_\varepsilon + \Delta W_\varepsilon + \tfrac{X}{2}.\nabla W_\varepsilon.$$

In this section, we obtain several energy estimates in various functions spaces. More precisely, assuming that T is large enough and W_0 is small enough in $\mathbb{H}^2(4)$, we show that the solution of (4.1) stays bounded in time in those energy spaces and is consequently global in time. In addition, we obtain the inequality (2.10) for W_ε.

We define $\Omega_\infty = \sum_{i=1}^{3} b_i f_i$, where $b_i = \int_{\mathbb{R}^3} p_i(X).W_0(X)dX$, and $\{f_1, f_2, f_3\}$ is the basis of the eigenspace of $\mathcal{L}$ associated to the eigenvalue -1, given by (2.1). The decomposition (2.3) becomes

$$W_\varepsilon(\tau) = e^{-\tau}\Omega_\infty + R_\varepsilon(\tau). \tag{4.2}$$

A short computation shows that R_ε satisfies the equality

$$\begin{aligned}\partial_\tau\left(R_\varepsilon - \alpha e^{-\tau}\Delta R_\varepsilon\right) + \varepsilon e^{-\tau}\Delta^2 R_\varepsilon - \mathcal{L}(R_\varepsilon) + \operatorname{curl}\ \left((W_\varepsilon - \alpha e^{-\tau}\Delta W_\varepsilon) \times U_\varepsilon\right)\\ +\alpha e^{-\tau}\Delta R_\varepsilon + \alpha e^{-\tau}\tfrac{X}{2}.\nabla\Delta R_\varepsilon + 3\alpha e^{-2\tau}\Delta\Omega_\infty + \varepsilon e^{-2\tau}\Delta^2\Omega_\infty = 0.\end{aligned} \tag{4.3}$$

In this section, we assume that W_0 satisfies the condition (2.9) for some positive constant γ. Let M be a positive constant such that $M \geq 2$ which will be made more precise later. We define τ_ε^* the largest positive time such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\begin{aligned}\|W_\varepsilon(\tau)\|_{L^2(4)}^2 + \|\nabla W_\varepsilon(\tau)\|_{L^2}^2 + \alpha e^{-\tau}\|\Delta W_\varepsilon(\tau)\|_{L^2}^2\\ +\alpha^2 e^{-2\tau}\left\||X|^4\Delta W_\varepsilon(\tau)\right\|_{L^2}^2 \leq M\gamma\left(\frac{3}{2} - \theta\right)^2.\end{aligned} \tag{4.4}$$

Since R_ε belongs to $C^0\left([\tau_0, \tau_\varepsilon^*), \mathbb{H}^2(4)\right)$, the time τ_ε^* is well defined. The next lemma gives two inequalities on ∇W_ε and R_ε.

Lemma 4.1 *Let $R_\varepsilon \in C^0\left([\tau_0, \tau_\varepsilon^*), \mathbb{H}^2(4)\right)$ satisfying the condition (4.4), there exists a positive constant C such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,*

$$\alpha e^{-\tau}\left\||X|^4\nabla W_\varepsilon(\tau)\right\|_{L^2}^2 \leq CM\gamma\left(\frac{3}{2} - \theta\right)^2. \tag{4.5}$$

Let $R_\varepsilon = W_\varepsilon - e^{-\tau}\Omega_\infty$. There exists a positive constant C such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\begin{aligned}|b|^2 + \|R_\varepsilon(\tau)\|_{L^2(4)}^2 + \|\nabla R_\varepsilon(\tau)\|_{L^2}^2 + \alpha e^{-\tau}\|\Delta R_\varepsilon(\tau)\|_{L^2}^2\\ +\alpha^2 e^{-2\tau}\left\||X|^4\Delta R_\varepsilon(\tau)\right\|_{L^2}^2 \leq CM\gamma\left(\tfrac{3}{2} - \theta\right)^2.\end{aligned} \tag{4.6}$$

Proof: The proof of the inequality (4.5) comes from an integration by parts. In fact, applying Hölder inequalities, one has

$$\begin{aligned}
\left\| |X|^4 \nabla W_\varepsilon \right\|_{L^2}^2 &= \int_{\mathbb{R}^3} |X|^8 \left|\nabla W_\varepsilon\right|^2 dX. \\
&= -\int_{\mathbb{R}^3} |X|^8 \Delta W_\varepsilon . W_\varepsilon dX - 8\int_{\mathbb{R}^3} |X|^6 X.\nabla W_\varepsilon . W_\varepsilon dX \\
&= -\int_{\mathbb{R}^3} |X|^8 \Delta W_\varepsilon . W_\varepsilon dX + 36\int_{\mathbb{R}^3} |X|^6 \left(W_\varepsilon\right)^2 dX \\
&\leq \left\| |X|^4 \Delta W_\varepsilon \right\|_{L^2} \left\| |X|^4 W_\varepsilon \right\|_{L^2} + 36 \left\| |X|^4 W_\varepsilon \right\|_{L^2}^{3/4} \left\| W_\varepsilon \right\|_{L^2}^{1/4}.
\end{aligned}$$

Due to the inequality (4.4), we obtain (4.5). To prove the inequality (4.6), we notice that , for all $i \in \{1,2,3\}$,

$$\begin{aligned}
|b_i| &\leq \int_{\mathbb{R}^2} |X| \left|W_0\right| dX \\
&\leq \left(\int_{\mathbb{R}^2} \frac{1}{\left(1+|X|^2\right)^3} dX\right)^{1/2} \left(\int_{\mathbb{R}^2} \left(1+|X|^2\right)^3 |X|^2 \left|W_0\right|^2 DX\right)^{1/2} \\
&\leq C \left\| W_0 \right\|_{L^2(4)}.
\end{aligned}$$

Thus, recalling that $R_\varepsilon = W_\varepsilon - e^{-\tau} \sum_{i=1}^{3} b_i f_i$ and taking into account (2.9), we obtain (4.6).

□

For sake of simplicity, we will assume in this section that $\gamma \leq 1$ and $\left(\frac{3}{2} - \theta\right) \leq 1$.

4.1 Estimates in $H^{-(\theta+2)}(\mathbb{R}^3)$

In this section, we perform an estimate of R_ε in the space $H^{-(\theta+2)}(\mathbb{R}^3)$ on the time interval $[\tau_0, \tau_\varepsilon^*)$. This operation is motivated by the H^1 estimate that we establish later. Indeed, in the H^1 estimate, we obtain terms involving the L^2 norm of R_ε that we cannot absorb directly. To overcome this problem, we look for an estimate in the homogeneous Sobolev space $\dot{H}^{-(\theta+2)}(\mathbb{R}^3)$. Combined with the next ones, it gives an estimate in the classical Sobolev space $H^{-(\theta+2)}(\mathbb{R}^3)$. To this end, we define, for $s \in \mathbb{R}$, the operator

$$(-\Delta)^{-s}u = \bar{\mathcal{F}}\left(\frac{1}{|\xi|^{4s}}\widehat{u}\right),$$

where $\widehat{u}$ is the Fourier transform of u, given by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix.\xi}u(x)dx,$$

and $\bar{\mathcal{F}}$ is the inverse Fourier transform.

In this section, given $0 \leq \theta < \frac{3}{2}$, we apply the linear operator $(-\Delta)^{-\left(\frac{\theta}{2}+1\right)}$ to (4.3) and then make the L^2-inner product of it with $(-\Delta)^{-\left(\frac{\theta}{2}+1\right)}R_\varepsilon$. We are allowed to consider $(-\Delta)^{-\left(\frac{\theta}{2}+1\right)}R_\varepsilon$ by the lemma

Lemma 4.2 *Let $u \in L^2(4)$ such that $\int_{\mathbb{R}^3} u(x)dx = 0$.*

1. *If $\int_{\mathbb{R}^3} x_i u(x)dx = 0$ for every $i \in \{1,2,3\}$, then, for all $0 \leq s < \frac{7}{4}$, $(-\Delta)^{-s}u \in L^2(\mathbb{R}^3)$ and there exists a positive constant C such that*
$$\left\|(-\Delta)^{-s}u\right\|_{L^2} \leq \frac{C}{\sqrt{7-4s}}\|u\|_{L^2(4)}. \tag{4.7}$$
2. *For all $0 \leq s < \frac{7}{4}$, $(-\Delta)^{-s}\nabla u \in L^2(\mathbb{R}^3)^3$ and there exists a positive constant C such that*
$$\left\|(-\Delta)^{-s}\nabla u\right\|_{L^2} \leq \frac{C}{\sqrt{7-4s}}\|u\|_{L^2(3)}. \tag{4.8}$$

Proof: Using Fourier variables, we get

$$\begin{aligned}\left\|(-\Delta)^{-s}u\right\|_{L^2}^2 &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^2}\frac{1}{|\xi|^{4s}}|\widehat{u}(\xi)|^2 d\xi \\ &\leq \frac{1}{(2\pi)^3}\int_{|\xi|\leq 1}\frac{1}{|\xi|^{4s}}|\widehat{u}(\xi)|^2 d\xi + \|u\|_{L^2}^2.\end{aligned}$$

We note $I = \frac{1}{(2\pi)^3}\int_{|\xi|\leq 1}\frac{1}{|\xi|^{4s}}|\widehat{u}(\xi)|^2 d\xi$. Using the fact that $\widehat{u}(0) = \int_{\mathbb{R}^3} u(x)dx = 0$ and the Cauchy-Schwartz inequality on $(0,1)$, we have

$$\begin{aligned}I &= \frac{1}{(2\pi)^3}\int_{|\xi|\leq 1}\frac{1}{|\xi|^{4s}}\left|\int_0^1 \xi.\nabla\widehat{u}(\sigma\xi)d\sigma\right|^2 d\xi \\ &\leq C\int_{|\xi|\leq 1}\frac{1}{|\xi|^{4s-2}}\int_0^1 |\nabla\widehat{u}(\sigma\xi)|^2 d\sigma d\xi.\end{aligned}$$

Then, due to the fact that $\partial_j \widehat{u}(0) = i \int_{\mathbb{R}^2} x_j u(x) dx = 0$, we get

$$\begin{aligned}
I &\leq C \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} \int_0^1 \left(\sum_{i,j=1}^{3} \left| \int_0^1 \xi_j \partial_i \partial_j \widehat{u}(r\sigma\xi) dr \right|^2 \right) d\sigma d\xi \\
&\leq C \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-4}} \int_0^1 \int_0^1 \left| \nabla^2 \widehat{u}(r\sigma\xi) \right|^2 dr d\sigma d\xi.
\end{aligned}$$

Finally, the continuous injection of $H^2(\mathbb{R}^3)$ into $L^\infty(\mathbb{R}^3)$ gives

$$\begin{aligned}
I &\leq \frac{C}{7-4s} \left\| \nabla^2 \widehat{u} \right\|_{L^\infty}^2 \\
&\leq \frac{C}{7-4s} \left\| \nabla^2 \widehat{u} \right\|_{H^2}^2 \\
&\leq \frac{C}{7-4s} \left\| u \right\|_{L^2(4)}^2 ,
\end{aligned}$$

and thus the inequality (4.7) is shown.

To get (4.8), using Fourier variables, we have

$$\begin{aligned}
\left\| (-\Delta)^{-s} \nabla u \right\|_{L^2}^2 &= \frac{1}{(2\pi)^3} \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} \left| \widehat{u}(\xi) \right|^2 d\xi + \left\| u \right\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^3} \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} \left| \int_0^1 \xi . \nabla \widehat{u}(s\xi) ds \right|^2 d\xi + \left\| u \right\|_{L^2}^2 \\
&\leq \frac{1}{(2\pi)^3} \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-4}} \left| \int_0^1 \left| \nabla \widehat{u}(s\xi) \right| ds \right|^2 d\xi + \left\| u \right\|_{L^2}^2 .
\end{aligned}$$

Using now Hölder inequalities, the fact that $4s - 4 < 3$ and the continuous injection of $H^2(\mathbb{R}^3)$ into $L^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}
\left\| (-\Delta)^{-s} \nabla u \right\|_{L^2}^2 &\leq C \int_0^1 \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-4}} \left| \nabla \widehat{u}(s\xi) \right|^2 d\xi ds + \left\| u \right\|_{L^2}^2 \\
&\leq C \left(\int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-4}} d\xi \right) \left\| \nabla \widehat{u} \right\|_{L^\infty}^2 + \left\| u \right\|_{L^2}^2 \\
&\leq \frac{C}{7-4s} \left\| u \right\|_{L^2(3)}^2 + \left\| u \right\|_{L^2}^2 .
\end{aligned}$$

□

In order to apply the lemma 4.2 to the non linear terms of the equation (4.3), we state the following lemma.

Lemma 4.3 *Let $w \in \mathbb{H}^2(4)$ and u obtained from w via the Biot-Savart law (2.15). For all $C \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}^3} (w(x) - C\Delta w(x)) \times u(x)dx = 0. \tag{4.9}$$

Proof: In order to show this equality, we just have to consider (2.8). An integration by parts gives directly (4.3).

□

Lemma 4.4 *Let u belongs to $H^2(4)$ and s such that $0 \le s < \frac{7}{4}$, then u satisfies the equalities*

1. $\left((-\Delta)^{-s}\mathcal{L}(u), (-\Delta)^{-s}u\right)_{L^2} = -\left\|(-\Delta)^{\frac{1}{2}-s}u\right\|_{L^2}^2 - \left(s - \frac{1}{4}\right)\left\|(-\Delta)^{-s}u\right\|_{L^2}^2.$
2. $\left((-\Delta)^{-s}\left(\frac{x}{2}.\nabla\Delta u\right), (-\Delta)^{-s}u\right)_{L^2} = \left(s + \frac{5}{4}\right)\left\|(-\Delta)^{\frac{1}{2}-s}u\right\|_{L^2}^2.$

This lemma is easily obtained with a few integrations by parts, when passing into Fourier variables.

In this section, to simplify the notations, we note R instead of R_ε, W instead of W_ε and U instead of U_ε. We also note U_∞, the divergence free vector field obtained from Ω_∞ via the Biot-Savart law and K the divergence free vector field obtained from R via the Biot-Savart law. We assume also, without loss of generality, that T is sufficiently large so that $\alpha e^{-\tau_0} \le 1$, where we recall that $\tau_0 = \log(T)$. We define the energy functional

$$E_0(\tau) = \frac{1}{2}\left(\left\|(-\Delta)^{-\left(\frac{\theta}{2}+1\right)}R\right\|_{L^2}^2 + \alpha e^{-\tau}\left\|(-\Delta)^{-\left(\frac{\theta+1}{2}\right)}R\right\|_{L^2}^2\right).$$

The next lemma gives a $H^{-(\theta+2)}$ which is necessary to obtain a good rate of convergence in the theorem 2.1. Actually, the space $H^{-(\theta+2)}$ is chosen to get the rate of convergence θ.

Lemma 4.5 *Let $W \in C^1\left((\tau_0, \tau_\varepsilon), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), \mathbb{H}^3(4)\right)$ be the solution of (4.1). There exist two positive constant γ_0 and T_0 such that, if $T \ge T_0$ and W_ε satisfy the* *condition (4.4) for some γ such that $0 < \gamma \leq \gamma_0$, then there exists a positive constant C such that*

$$\begin{aligned} \partial_\tau E_0 + 2\theta E_0 + \frac{1}{2} \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 \leq \\ CM\gamma \left(\left\| |X|^4 R \right\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2(4)}^2 \right) + CM\gamma e^{-4\tau}. \end{aligned} \tag{4.10}$$

Proof: To prove this lemma, we apply the operator $(-\Delta)^{-\left(\frac{\theta}{2}+1\right)}$ to (4.3) and make the L^2-inner product of it with $(-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R$. Applying Lemma 4.4 and through some easy computations, one has

$$\begin{aligned} &\frac{1}{2}\partial_\tau \left(\left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \alpha e^{-\tau} \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 \right) + \varepsilon e^{-\tau} \left\| (-\Delta)^{-\frac{\theta}{2}} R \right\|_{L^2}^2 \\ &+ \left(\frac{\theta}{2} + \frac{3}{4} \right) \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \left(1 + \left(\frac{\theta}{2} + \frac{3}{4} \right) \alpha e^{-\tau} \right) \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 = I_1 + I_2, \end{aligned} \tag{4.11}$$

where

$$I_1 = \left((-\Delta)^{-\left(\frac{\theta}{2}+1\right)} \left(\text{curl } \left(U \times \nabla \left(W - \alpha e^{-\tau} \Delta W \right) \right) \right), (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right)_{L^2},$$

$$I_2 = e^{-2\tau} \left((-\Delta)^{-\left(\frac{\theta}{2}+1\right)} \left(-\alpha \Delta \Omega_\infty - \varepsilon \Delta^2 \Omega_\infty \right), (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right)_{L^2}.$$

We start with the estimate of the easiest term, that is I_2. Using Cauchy-Schwartz inequality, we get

$$\begin{aligned} I_2 \leq \alpha e^{-2\tau} \left\| (-\Delta)^{-\frac{\theta}{2}} \Omega_\infty \right\|_{L^2} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ + \varepsilon e^{-2\tau} \left\| (-\Delta)^{1-\frac{\theta}{2}} \Omega_\infty \right\|_{L^2} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}. \end{aligned}$$

Using Lemma 4.2 and taking into account the good regularity of Ω_∞ and the inequality (4.6), one has

$$\begin{aligned} I_2 &\leq Ce^{-2\tau} \|\Omega_\infty\|_{H^2(4)} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ &\leq \mu \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \frac{C|b|^2}{\mu} e^{-4\tau} \\ &\leq \mu \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta \right)^2}{\mu} e^{-4\tau}, \end{aligned} \tag{4.12}$$

where μ is a positive constant that will be made more precise later.

It remains to bound I_1. Using Cauchy-Schwartz inequality and the lemmas 4.3 and 4.2, we obtain

$$\begin{aligned} I_1 &\leq C \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} \nabla \left(U \times \left(W - \alpha e^{-\tau} \Delta W \right) \right) \right\|_{L^2} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ &\leq \frac{C}{\left(\frac{3}{2}-\theta\right)^{1/2}} \left\| U \left(W - \alpha e^{-\tau} \Delta W \right) \right\|_{L^2(4)} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ &\leq \frac{C}{\left(\frac{3}{2}-\theta\right)^{1/2}} \| U \|_{L^\infty} \left\| W - \alpha e^{-\tau} \Delta W \right\|_{L^2(4)} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}. \end{aligned}$$

The inequality (2.17) of the lemma 2.2 with $p = 2$, $q = 6$ and $\eta = \frac{1}{2}$ and the continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$ yield

$$\begin{aligned} I_1 &\leq \frac{C}{\left(\frac{3}{2}-\theta\right)^{1/2}} \| W \|_{L^2}^{1/2} \| W \|_{L^6}^{1/2} \left\| W - \alpha e^{-\tau} \Delta W \right\|_{L^2(4)} \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ &\leq \frac{C}{\left(\frac{3}{2}-\theta\right)^{1/2}} \| W \|_{H^1} \left(\| W \|_{L^2(4)} + \alpha e^{-\tau} \| \Delta W \|_{L^2(4)} \right) \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2} \\ &\leq \mu \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 \\ &\qquad + \frac{C}{\mu \left(\frac{3}{2}-\theta\right)} \left(\| W \|_{L^2}^2 + \| \nabla W \|_{L^2}^2 \right) \left(\| W \|_{L^2(4)}^2 + \alpha^2 e^{-2\tau} \| \Delta W \|_{L^2(4)}^2 \right). \end{aligned}$$

We use now the decomposition (4.2) to get

$$\begin{aligned} I_1 \leq\ & \mu \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 \\ & + \frac{C}{\mu \left(\frac{3}{2}-\theta\right)} \left(\| R \|_{L^2}^2 + \| \nabla R \|_{L^2}^2 \right) \left(\| W \|_{L^2(4)}^2 + \alpha^2 e^{-2\tau} \| \Delta W \|_{L^2(4)}^2 \right) \\ & + \frac{C e^{-2\tau}}{\mu \left(\frac{3}{2}-\theta\right)} \left(\| \Omega_\infty \|_{L^2}^2 + \| \nabla \Omega_\infty \|_{L^2}^2 \right) \left(\| R \|_{L^2(4)}^2 + \alpha^2 e^{-2\tau} \| \Delta R \|_{L^2(4)}^2 \right) \\ & + \frac{C e^{-4\tau}}{\mu \left(\frac{3}{2}-\theta\right)} \left(\| \Omega_\infty \|_{L^2}^2 + \| \nabla \Omega_\infty \|_{L^2}^2 \right) \left(\| \Omega_\infty \|_{L^2(4)}^2 + \alpha^2 e^{-2\tau} \| \Delta \Omega_\infty \|_{L^2(4)}^2 \right). \end{aligned}$$

Finally, using the inequalities (4.4) and (4.6) we get

$$\begin{aligned} I_1 \leq \mu \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \frac{CM\left(\frac{3}{2}-\theta\right)\gamma e^{-4\tau}}{\mu} \\ + \frac{CM\gamma\left(\frac{3}{2}-\theta\right)}{\mu}\left(\|R\|_{L^2(4)}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2(4)}^2 \right). \end{aligned} \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), it comes

$$\begin{aligned} &\frac{1}{2}\partial_\tau \left(\left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 + \alpha e^{-\tau} \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 \right) + \varepsilon e^{-\tau} \left\| (-\Delta)^{-\frac{\theta}{2}} R \right\|_{L^2}^2 \\ &\qquad + \left(\theta + \frac{1}{2}\left(\frac{3}{2} - \theta - 2\mu \right)\right) \left\| (-\Delta)^{-\left(\frac{\theta}{2}+1\right)} R \right\|_{L^2}^2 \\ &\qquad + \left(1 + \left(\frac{\theta}{2} + \frac{3}{4} \right) \alpha e^{-\tau} \right) \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 \\ &\leq \frac{CM\gamma\left(\frac{3}{2}-\theta\right)}{\mu}\left(\|R\|_{L^2(4)}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2(4)}^2 \right) + \frac{CM\left(\frac{3}{2}-\theta\right)\gamma e^{-4\tau}}{\mu}. \end{aligned} \tag{4.14}$$

We set $\mu = \dfrac{\frac{3}{2}-\theta}{2}$, and we obtain

$$\begin{aligned} \partial_\tau E_0 + 2\theta E_0 + \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^2 \leq \\ CM\gamma \left(\|R\|_{L^2(4)}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2(4)}^2 \right) + CM\gamma e^{-4\tau}. \end{aligned} \tag{4.15}$$

Furthermore, using Fourier variables and Hölder inequalities, we see that

$$\begin{aligned} \|R\|_{L^2}^2 &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left| \widehat{R}(\xi) \right|^2 d\xi \\ &\leq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\xi|^{\frac{2(1+\theta)}{2+\theta}} \left| \widehat{R}(\xi) \right|^{\frac{2(1+\theta)}{2+\theta}} \frac{1}{|\xi|^{\frac{2(1+\theta)}{2+\theta}}} \left| \widehat{R}(\xi) \right|^{\frac{2}{2+\theta}} d\xi \\ &\leq \left(\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{|\xi|^{2(\theta+1)}} \left| \widehat{R}(\xi) \right|^2 d\xi \right)^{\frac{1+\theta}{2+\theta}} \left(\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\xi|^2 \left| \widehat{R}(\xi) \right|^2 d\xi \right)^{\frac{1}{2+\theta}} \\ &\leq \left\| (-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R \right\|_{L^2}^{\frac{2(1+\theta)}{2+\theta}} \|\nabla R\|_{L^2}^{\frac{2}{2+\theta}}. \end{aligned}$$

Using a convexity inequality, it is easy to see that

$$\|R\|_{L^2}^2 \leq \frac{1}{\eta^{\frac{2+\theta}{1+\theta}}} \left(\frac{1+\theta}{2+\theta}\right) \left\|(-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R\right\|_{L^2}^2 + \frac{\eta^{2+\theta}}{2+\theta} \|\nabla R\|_{L^2}^2 ,$$

for all $0 < \eta \leq 1$.

Via a short computation, using the fact that $0 < \theta < \frac{3}{2}$ and $0 < \eta \leq 1$, one obtains

$$\|R\|_{L^2}^2 \leq \frac{5}{7\eta^2} \left\|(-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R\right\|_{L^2}^2 + \frac{\eta^2}{2} \|\nabla R\|_{L^2}^2 . \tag{4.16}$$

Applying (4.16) with $\eta = 1$ and taking γ small enough, the inequality (4.15) becomes

$$\begin{aligned} \partial_\tau E_0 + 2\theta E_0 + \frac{1}{2} \left\|(-\Delta)^{-\left(\frac{\theta+1}{2}\right)} R\right\|_{L^2}^2 \leq \\ CM\gamma \left(\left\||X|^4 R\right\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2(4)}^2\right) + CM\gamma e^{-4\tau}. \end{aligned} \tag{4.17}$$

$\square$

4.2 Estimates in $H^1(\mathbb{R}^3)$

This section is devoted to the H^1 estimate of the solutions of (4.3) under the condition (4.4). In particular, we see in this section that the previous estimate in $\dot{H}^{-(1+\theta)}$ enables to absorb the terms involving the L^2-norm of R. To obtain this H^1 estimate, we make the L^2-scalar product of (4.3) with R. We define the energy functional

$$E_1(\tau) = \frac{1}{2}\left(\|R\|_{L^2}^2 + \alpha e^{-\tau} \|\nabla R\|_{L^2}^2\right).$$

The estimate of R in the Sobolev space $H^1(\mathbb{R}^3)$ is given by the next lemma.

Lemma 4.6 *Let $W \in C^1\left((\tau_0, \tau_\varepsilon), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), \mathbb{H}^3(4)\right)$ be the solution of (4.1). There exist two positive constants γ_0 and T_0 such that, if $T \geq T_0$ and W satisfy the condition (4.4) for some γ such that $0 < \gamma \leq \gamma_0$, then there exists a positive constant C such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,*

$$\begin{aligned} \partial_\tau E_1 + 3E_1 + \frac{1}{2} \|\nabla R\|_{L^2}^2 \leq \frac{7}{4} \|R\|_{L^2}^2 + CM^2\gamma \left(\frac{3}{2} - \theta\right)^2 e^{-4\tau} \\ + CM\gamma \left(\frac{3}{2} - \theta\right)^2 \left(\|R\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2}^2\right). \end{aligned} \tag{4.18}$$

Proof: We make the L^2-scalar product of (4.3) with R. Integrating several times by parts, we obtain

$$\frac{1}{2}\partial_\tau\left(\|R\|_{L^2}^2+\alpha e^{-\tau}\|\nabla R\|_{L^2}^2\right)+\varepsilon\|\Delta R\|_{L^2}^2+\left(1-\frac{\alpha}{4}e^{-\tau}\right)\|\nabla R\|_{L^2}^2-\frac{1}{4}\|R\|_{L^2}^2=I_1+I_2, \tag{4.19}$$

where

$$I_1=\left(\text{curl }\left(U\times\left(W-\alpha e^{-\tau}\Delta W\right)\right),R\right)_{L^2},$$

$$I_2=e^{-2\tau}\left(-\alpha\Delta\Omega_\infty-\varepsilon\Delta^2\Omega_\infty,R\right)_{L^2}.$$

As usual, because of the good regularity of Ω_∞, the easiest term to estimate is I_2. Integrating by parts, one has

$$I_2=e^{-2\tau}\left(\alpha\nabla\Omega_\infty+\varepsilon\nabla\Delta\Omega_\infty,\nabla R\right)_{L^2}.$$

Using Hölder and Young inequalities and the inequality (4.6), we get

$$\begin{aligned}I_2 &\leq e^{-2\tau}\left(\alpha\|\nabla\Omega_\infty\|_{L^2}+\varepsilon\|\nabla\Delta\Omega_\infty\|_{L^2}\right)\|\nabla R\|_{L^2}\\ &\leq C\,|b|\,(\alpha+\varepsilon)\,e^{-2\tau}\|\nabla R\|_{L^2}\\ &\leq \mu\|\nabla R\|_{L^2}^2+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau},\end{aligned} \tag{4.20}$$

where μ is a positive constant that will be made more precise later.

The last remaining term will be estimated by the same way, using the divergence free property of U. Integrating by parts, we obtain

$$I_1=\left(U\times\left(W-\alpha e^{-\tau}\Delta W\right),\text{curl }R\right)_{L^2}.$$

We decompose I_1 as the sum of three terms

$$I_1=I_1^1+I_1^2+I_1^3,$$

where

$$I_1^1=\left(K\times\left(W-\alpha e^{-\tau}\Delta W\right),\text{curl }R\right)_{L^2},$$

$$I_1^2=e^{-\tau}\left(V_\infty\times\left(R-\alpha e^{-\tau}\Delta R\right),\text{curl }R\right)_{L^2},$$

$$I_1^3=e^{-2\tau}\left(V_\infty\times\left(\Omega_\infty-\alpha e^{-\tau}\Delta\Omega_\infty\right),\text{curl }R\right)_{L^2}.$$

Hölder inequalities lead to

$$\begin{aligned}
I_1^1 &\leq C\left(\|KW\|_{L^2}+\alpha e^{-\tau}\|K\Delta W\|_{L^2}\right)\|\nabla R\|_{L^2}\\
&\leq C\|K\|_{L^\infty}\left(\|W\|_{L^2}+\alpha e^{-\tau}\|\Delta W\|_{L^2}\right)\|\nabla R\|_{L^2}.
\end{aligned}$$

Applying the inequality (2.17) with $p=2$, $q=6$ and $\eta=\frac{1}{2}$ and using the continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, one gets

$$\begin{aligned}
I_1^1 &\leq C\|R\|_{L^2}^{1/2}\|R\|_{L^6}^{1/2}\left(\|W\|_{L^2}+\alpha e^{-\tau}\|\Delta W\|_{L^2}\right)\|\nabla R\|_{L^2}\\
&\leq C\|R\|_{L^2}^{1/2}\|R\|_{H^1}^{1/2}\left(\|W\|_{L^2}+\alpha e^{-\tau}\|\Delta W\|_{L^2}\right)\|\nabla R\|_{L^2}.
\end{aligned}$$

Then, we use Young inequality and the inequality (4.4). We obtain

$$\begin{aligned}
I_1^1 &\leq \mu\|\nabla R\|_{L^2}^2+\frac{C}{\mu}\left(\|W\|_{L^2}^2+\alpha^2e^{-2\tau}\|\Delta W\|_{L^2}^2\right)\left(\|R\|_{L^2}^2+\|\nabla R\|_{L^2}^2\right)\\
&\leq \mu\|\nabla R\|_{L^2}^2+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}\left(\|R\|_{L^2}^2+\|\nabla R\|_{L^2}^2\right).
\end{aligned}$$

Hölder inequalities yield

$$I_1^2 \leq Ce^{-\tau}\|V_\infty\|_{L^\infty}\left(\|R\|_{L^2}+\alpha e^{-\tau}\|\Delta R\|_{L^2}\right)\|\nabla R\|_{L^2}.$$

Applying the inequality (2.17) of the lemma 2.2 with $p=2$, $q=6$ and $\eta=\frac{1}{2}$, and the inequality (4.6), we get

$$\begin{aligned}
I_1^2 &\leq Ce^{-\tau}\|\Omega_\infty\|_{L^2}^{1/2}\|\Omega_\infty\|_{L^6}^{1/2}\left(\|R\|_{L^2}+\alpha e^{-\tau}\|\Delta R\|_{L^2}\right)\|\nabla R\|_{L^2}\\
&\leq C|b|e^{-\tau}\left(\|R\|_{L^2}+\alpha e^{-\tau}\|\Delta R\|_{L^2}\right)\|\nabla R\|_{L^2}\\
&\leq \mu\|\nabla R\|_{L^2}^2+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}\left(\|R\|_{L^2}^2+\alpha^2e^{-2\tau}\|\Delta R\|_{L^2}^2\right).
\end{aligned}$$

It remains to estimate I_1^3. By the same computations, we get

$$\begin{aligned}
I_1^3 &\leq \mu\|\nabla R\|_{L^2}^2+\frac{C}{\mu}e^{-4\tau}\|V_\infty\|_{L^\infty}^2\left(\|\Omega_\infty\|_{L^2}^2+\alpha^2e^{-2\tau}\|\Delta\Omega_\infty\|_{L^2}^2\right)\\
&\leq \mu\|\nabla R\|_{L^2}^2+\frac{C}{\mu}e^{-4\tau}\|\Omega_\infty\|_{L^2}\|\Omega_\infty\|_{L^6}\left(\|\Omega_\infty\|_{L^2}^2+\alpha^2e^{-2\tau}\|\Delta\Omega_\infty\|_{L^2}^2\right)\\
&\leq \mu\|\nabla R\|_{L^2}^2+\frac{CM^2\gamma^2\left(\frac{3}{2}-\theta\right)^4}{\mu}e^{-4\tau}.
\end{aligned}$$

In particular, we have shown that

$$\begin{aligned} I_1 \le 3\mu \|\nabla R\|_{L^2}^2 &+ \frac{CM^2\gamma^2\left(\frac{3}{2}-\theta\right)^4}{\mu} e^{-4\tau} \\ &+ \frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right). \end{aligned} \tag{4.21}$$

Thus, due to the inequalities (4.20) and (4.21), the inequality (4.19) becomes

$$\begin{aligned} \partial_\tau E_1 + 3E_1 + \left(1 - 4\mu - \frac{7\alpha}{4}e^{-\tau}\right)\|\nabla R\|_{L^2}^2 \le \frac{7}{4}\|R\|_{L^2}^2 + \frac{CM^2\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau} \\ + \frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right). \end{aligned} \tag{4.22}$$

Taking γ_0 and μ small enough and $T = e^{\tau_0}$ large enough, we obtain

$$\begin{aligned} \partial_\tau E_1 + 3E_1 + \frac{1}{2}\|\nabla R\|_{L^2}^2 \le \frac{7}{4}\|R\|_{L^2}^2 + CM\gamma\left(\frac{3}{2}-\theta\right)^2\left(\|R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right) \\ + CM^2\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-4\tau}. \end{aligned} \tag{4.23}$$

$\square$

Using the interpolation inequality (4.16), we get

$$\begin{aligned} \partial_\tau E_1 + 3E_1 + \frac{1}{2}\|\nabla R\|_{L^2}^2 \le \frac{7}{4}\left(\frac{5}{7\eta^2}\left\|(-\Delta)^{-\left(\frac{1+\theta}{2}\right)}R\right\|_{L^2}^2 + \frac{\eta^2}{2}\|\nabla R\|_{L^2}^2\right) \\ + CM\gamma\left(\tfrac{3}{2}-\theta\right)^2\left(\left\|(-\Delta)^{-\left(\frac{1+\theta}{2}\right)}R\right\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right) \\ + CM^2\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-4\tau}, \end{aligned} \tag{4.24}$$

where $0 < \eta \le 1$.

Taking $\eta = \sqrt{\frac{2}{7}}$ and γ sufficiently small, we get

$$\partial_\tau E_1 + 3E_1 + \frac{1}{4}\|\nabla R\|_{L^2}^2 \leq \left(\frac{35}{8} + CM\gamma\left(\frac{3}{2}-\theta\right)^2\right)\left\|(-\Delta)^{-\left(\frac{1+\theta}{2}\right)}R\right\|_{L^2}^2$$
$$+CM\gamma\left(\tfrac{3}{2}-\theta\right)^2\left(\left\|(-\Delta)^{-\left(\frac{1+\theta}{2}\right)}R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right) + CM^2\gamma\left(\tfrac{3}{2}-\theta\right)^2 e^{-4\tau}. \tag{4.25}$$

Using the two energies E_0 and E_1, we define

$$E_2 = 6E_0 + E_1.$$

Combining the inequalities (4.10) and (4.25) and setting γ sufficiently small, we check that

$$\begin{aligned}\partial E_2(\tau) + 2\theta E_2(\tau) + \left\|(-\Delta)^{-\left(\frac{1+\theta}{2}\right)}R\right\|_{L^2}^2 + \frac{1}{4}\|\nabla R\|_{L^2}^2 \leq \\ CM\gamma\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\|\Delta R\|_{L^2}^2\right) + CM^2\gamma e^{-4\tau}.\end{aligned} \tag{4.26}$$

4.3 Estimates in $H^2(\mathbb{R}^3)$

In this part, we perform an H^2 estimate for the solution R of (4.3) under the smallness assumption (4.4). To this end, we make the L^2-scalar product of (4.3) with $-\Delta R$. We define the functional

$$E_3(\tau) = \tfrac{1}{2}\left(\|\nabla R\|_{L^2}^2 + \alpha e^{-\tau}\|\Delta R\|_{L^2}^2\right).$$

The next lemma gives an estimate in the space $H^2(\mathbb{R}^3)$.

Lemma 4.7 *Let $W \in C^1\left((\tau_0,\tau_\varepsilon), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0,\tau_\varepsilon), \mathbb{H}^3(4)\right)$ be the solution of (4.1). There exist two positive constants γ_0 and T_0 such that, if $T \geq T_0$ and W satisfy the condition (4.4) for some positive constant γ such that $\gamma \leq \gamma_0$, then there exists $C > 0$ such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,*

$$\partial_\tau E_3 + 3E_3 + \frac{1}{2}\|\Delta R\|_{L^2}^2 \leq \frac{9}{4}\|\nabla R\|_{L^2}^2 + CM\gamma\left(\frac{3}{2}-\theta\right)^2\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2\right)$$
$$+CM^2\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-\frac{7\tau}{2}}. \tag{4.27}$$

Proof: The proof of Lemma 4.7 is made through the L^2-scalar product of (4.3) with $-\Delta R$. First of all, we remark that

$$\text{curl }\left(\left(W-\alpha e^{-\tau}\Delta W\right)\times U\right)=U.\nabla\left(W-\alpha e^{-\tau}\Delta W\right)-\left(W-\alpha e^{-\tau}\Delta W\right).\nabla U.$$

Making some computations that we let to the reader involving integrations by parts and the divergence free property of U, we obtain

$$\partial_\tau\left(\|\nabla R\|_{L^2}^2+\alpha e^{-\tau}\|\Delta R\|_{L^2}^2\right)+\left(1-\frac{3\alpha}{4}e^{-\tau}\right)\|\Delta R\|_{L^2}^2=\frac{3}{4}\|\nabla R\|_{L^2}^2+I_1+I_2+I_3, \quad (4.28)$$

where

$$I_1=\left(-U.\nabla\left(W-\alpha e^{-\tau}\Delta W\right),\Delta R\right)_{L^2},$$

$$I_2=\left(\left(W-\alpha e^{-\tau}\Delta W\right).\nabla U,\Delta R\right)_{L^2},$$

$$I_3=e^{-2\tau}\left(\alpha\Delta\Omega_\infty+\varepsilon\Delta^2\Omega_\infty,\Delta R\right)_{L^2}.$$

Like in the previous estimates, the easiest term is I_3. Indeed, using Hölder and Young inequalities and the inequality (4.6), one has

$$\begin{aligned} I_3 &\leq e^{-2\tau}\left(\alpha\|\Delta\Omega_\infty\|_{L^2}+\varepsilon\left\|\Delta^2\Omega_\infty\right\|_{L^2}\right)\|\Delta R\|_{L^2}\\ &\leq \mu\|\Delta R\|^2+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau}, \end{aligned} \quad (4.29)$$

where μ is a positive constant which will be made more precise later.

We now look for an estimate of I_1. We decompose it as follows:

$$I_1=I_1^1+I_1^2+I_1^3,$$

where

$$I_1^1=e^{-\tau}\left(K.\nabla\left(\Omega_\infty-\alpha e^{-\tau}\Delta\Omega_\infty\right),\Delta R\right)_{L^2},$$

$$I_1^2=e^{-2\tau}\left(V_\infty.\nabla\left(\Omega_\infty-\alpha e^{-\tau}\Delta\Omega_\infty\right),\Delta R\right)_{L^2},$$

$$I_1^3=\left(U.\nabla\left(R-\alpha e^{-\tau}\Delta R\right),\Delta R\right)_{L^2}.$$

Due to the smoothness of Ω_∞ and the inequality (2.17), we get

$$\begin{aligned} I_1^1 &\leq e^{-\tau} \|K\|_{L^\infty} \left(\|\nabla \Omega_\infty\|_{L^2} + \alpha e^{-\tau} \|\nabla \Delta \Omega_\infty\|_{L^2} \right) \|\Delta R\|_{L^2} \\ &\leq C\,|b|\, e^{-\tau} \|R\|_{L^2}^{1/2} \|R\|_{L^6}^{1/2} \|\Delta R\|_{L^2}. \end{aligned}$$

The continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, Young inequality and the inequality (4.6) yield

$$\begin{aligned} I_1^1 &\leq C\,|b|\, e^{-\tau} \|R\|_{H^1} \|\Delta R\|_{L^2} \\ &\leq \mu \|\Delta R\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} e^{-2\tau} \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \right). \end{aligned}$$

Doing the same computations, we get

$$I_1^2 \leq \mu \|\Delta R\|_{L^2}^2 + \frac{CM^2\gamma^2 \left(\frac{3}{2} - \theta\right)^4}{\mu} e^{-4\tau}.$$

The divergence free property of U and an integration by parts imply

$$I_1^3 = (U.\nabla R, \Delta R)_{L^2}.$$

Thus, using Hölder and Young inequalities, the lemma 2.2 and the inequality (4.4), we obtain,

$$\begin{aligned} I_1^3 &\leq \|U\|_{L^\infty} \|\nabla R\|_{L^2} \|\Delta R\|_{L^2} \\ &\leq C \|W\|_{L^2}^{1/2} \|W\|_{L^6}^{1/2} \|\nabla R\|_{L^2} \|\Delta R\|_{L^2} \\ &\leq C \|W\|_{H^1} \|\nabla R\|_{L^2} \|\Delta R\|_{L^2} \\ &\leq \mu \|\Delta R\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} \|\nabla R\|_{L^2}^2. \end{aligned}$$

Consequently, we have shown that

$$I_1 \leq 3\mu \|\Delta R\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \right) + \frac{CM^2\gamma^2 \left(\frac{3}{2} - \theta\right)^4}{\mu} e^{-4\tau}. \quad (4.30)$$

It remains to estimate I_2. We set

$$I_2 = I_2^1 + I_2^2,$$

where

$$I_2^1 = -\left(W.\nabla U, \Delta R\right)_{L^2},$$

$$I_2^2 = \alpha e^{-\tau}\left(\Delta W.\nabla U, \Delta R\right)_{L^2}.$$

Recalling that $W = e^{-\tau}\Omega_\infty + R$ and using Hölder and Young inequalities and the inequality (2.18) with $p = 4$, one has

$$\begin{aligned}
I_2^1 &\leq \|W\|_{L^4}\|\nabla U\|_{L^4}\|\Delta R\|_{L^2} \\
&\leq C\|W\|_{L^4}^2\|\Delta R\|_{L^2} \\
&\leq \mu\|\Delta R\|_{L^2}^2 + \frac{C}{\mu}\|W\|_{L^4}^4 \\
&\leq \mu\|\Delta R\|_{L^2}^2 + \frac{C}{\mu}\left(e^{-4\tau}\|\Omega_\infty\|_{L^4}^4 + \|R\|_{L^4}^4\right).
\end{aligned}$$

The condition (4.6) and the continuous injection of $H^1(\mathbb{R}^3)$ into $L^4(\mathbb{R}^3)$ yield

$$\begin{aligned}
I_2^1 &\leq \mu\|\Delta R\|_{L^2}^2 + \frac{CM^2\gamma^2\left(\frac{3}{2}-\theta\right)^4}{\mu}e^{-4\tau} + \frac{C}{\mu}\|R\|_{H^1}^4 \\
&\leq \mu\|\Delta R\|_{L^2}^2 + \frac{CM^2\gamma^2\left(\frac{3}{2}-\theta\right)^4}{\mu}e^{-4\tau} + \frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2\right).
\end{aligned}$$

Using the inequality (2.17) with $p = 2$, $q = 6$ and $\eta = \frac{1}{2}$ and the continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
I_2^2 &\leq \alpha e^{-\tau}\left(\|\Delta R\|_{L^2} + e^{-\tau}\|\Delta\Omega_\infty\|_{L^2}\right)\|\nabla U\|_{L^\infty}\|\Delta R\|_{L^2} \\
&\leq C\alpha e^{-\tau}\left(\|\Delta R\|_{L^2} + e^{-\tau}\|\Delta\Omega_\infty\|_{L^2}\right)\|\nabla W\|_{L^2}^{1/2}\|\nabla W\|_{L^6}^{1/2}\|\Delta R\|_{L^2} \\
&\leq C\alpha e^{-\tau}\left(\|\Delta R\|_{L^2} + e^{-\tau}\|\Delta\Omega_\infty\|_{L^2}\right)\|\nabla W\|_{L^2}^{1/2}\|W\|_{H^2}^{1/2}\|\Delta R\|_{L^2}.
\end{aligned}$$

We set $\delta = M\gamma\left(\frac{3}{2}-\theta\right)^2$. Taking into account the inequalities (4.6) and (4.4), it comes,

$$\begin{aligned}
I_2^2 &\leq C\delta^{1/2}e^{-\frac{3\tau}{4}}\left(\|\Delta R\|_{L^2} + \delta^{1/2}e^{-\tau}\right)\|\Delta R\|_{L^2} \\
&\leq C\delta^{1/2}e^{-\frac{3\tau}{4}}\|\Delta R\|_{L^2}^2 + C\delta e^{-\frac{7\tau}{4}}\|\Delta R\|_{L^2} \\
&\leq C\left(\delta^{1/2}e^{-\frac{3\tau}{4}} + \delta\right)\|\Delta R\|_{L^2}^2 + C\delta e^{-\frac{7\tau}{2}} \\
&\leq CM\gamma^{1/2}\left(\frac{3}{2}-\theta\right)\|\Delta R\|_{L^2}^2 + CM\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-\frac{7\tau}{2}}.
\end{aligned}$$

Finally, we have shown,

$$I_2 \leq \left(CM\gamma^{1/2} \left(\frac{3}{2} - \theta \right) + \mu \right) \|\Delta R\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \right) + \frac{CM^2\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} e^{-\frac{7\tau}{2}}. \tag{4.31}$$

Combining (4.28), (4.29), (4.30) and (4.31), one has

$$\partial_\tau E_3 + 3E_3 + \left(1 - 5\mu - \frac{9\alpha}{4} e^{-\tau} \right) \|\Delta R\|_{L^2}^2 \leq \frac{9}{4} \|\nabla R\|_{L^2}^2 + CM\gamma^{1/2} \left(\frac{3}{2} - \theta \right) \|\Delta R\|_{L^2}^2 + \frac{CM\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \right) + \frac{CM^2\gamma \left(\frac{3}{2} - \theta\right)^2}{\mu} e^{-\frac{7\tau}{2}}. \tag{4.32}$$

We take γ_0 and μ small enough and $T = e^{\tau_0}$ large enough compared to α and obtain

$$\partial_\tau E_3 + 3E_3 + \frac{1}{2} \|\Delta R\|_{L^2}^2 \leq \frac{9}{4} \|\nabla R\|_{L^2}^2 + CM\gamma \left(\frac{3}{2} - \theta \right)^2 \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \right) + CM^2\gamma \left(\frac{3}{2} - \theta \right)^2 e^{-\frac{7\tau}{2}}. \tag{4.33}$$

□

To achieve the H^2 estimate, we use E_2 and E_3 to define the functional

$$E_4 = 12E_2 + E_3.$$

Taking into account the two inequalities (4.26) and (4.27), we see that E_4 satisfies

$$\partial_\tau E_4 + 2\theta E_4 + 12 \left\| (-\Delta)^{-\left(\theta - \frac{1}{4}\right)} R \right\|_{L^2}^2 + \frac{3}{4} \|\nabla R\|_{L^2}^2 + \frac{1}{2} \|\Delta R\|_{L^2}^2 \leq + CM\gamma \left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \left\| |X|^4 R \right\|_{L^2}^2 + \alpha^2 e^{-2\tau} \|\Delta R\|_{L^2}^2 \right) + CM^2\gamma e^{-\frac{7\tau}{2}}. \tag{4.34}$$

Using again the interpolation inequality (4.16) and taking γ_0 small enough, this inequality becomes

$$\partial_\tau E_4 + 2\theta E_4 + 10 \left\| (-\Delta)^{-\left(\theta - \frac{1}{4}\right)} R \right\|_{L^2}^2 + \tfrac{1}{2} \|\nabla R\|_{L^2}^2 + \tfrac{1}{4} \|\Delta R\|_{L^2}^2 \leq CM\gamma \left\| |X|^4 R \right\|_{L^2}^2 + CM^2\gamma e^{-\frac{7\tau}{2}}. \tag{4.35}$$

4.4 Estimates in $H^2(4)$

To finish the energy estimates, we have to work in weighted spaces. We can see that the terms of the right hand side of the inequality (4.35) involve weighted L^2 norms that we have to absorb. In order to perform estimates in weighted Lebesgue norms, and additionally absorb the weighted terms of (4.35), we make the L^2-inner product of (4.3) with $|X|^8 \left(R - \alpha e^{-\tau} \Delta R\right)$. One defines the energy functional

$$E_5 = \frac{1}{2} \left\| |X|^4 \left(R - \alpha e^{-\tau} \Delta R\right) \right\|_{L^2}^2 .$$

The next lemma summarizes the terms provided by the linear part of (4.3), when making the L^2-scalar product with $|X|^8 \left(R - \alpha e^{-\tau} \Delta R\right)$.

Lemma 4.8 *Let u be a divergence free vector field of $\mathbb{H}^2(4)$, $a \in \mathbb{R}$ and $F(u) = |x|^8 (u - a\Delta u)$. The five next equalities hold.*

1.
$$\left(\Delta u, F(u)\right)_{L^2} = 36 \left\| |x|^3 u \right\|_{L^2}^2 - \left\| |x|^4 \nabla u \right\|_{L^2}^2 - a \left\| |x|^4 \Delta u \right\|_{L^2}^2 . \tag{4.36}$$

2.
$$\left(\frac{x}{2}.\nabla u, F(u)\right)_{L^2} = -\frac{11}{4} \left\| |x|^4 u \right\|_{L^2}^2 - \frac{9a}{4} \left\| |x|^4 \nabla u \right\|_{L^2}^2 + 4a \left\| |x|^3 (x.\nabla u) \right\|_{L^2}^2 . \tag{4.37}$$

3.
$$\begin{aligned}\left(\mathcal{L}(u), F(u)\right)_{L^2} = -\frac{7}{4} \left\| |x|^4 u \right\|_{L^2}^2 - \left(1 + \frac{5a}{4}\right) \left\| |x|^4 \nabla u \right\|_{L^2}^2 - a \left\| |x|^4 \Delta u \right\|_{L^2}^2 \\ +4a \left\| |x|^3 (x.\nabla u) \right\|_{L^2}^2 + 36 (1-a) \left\| |x|^3 u \right\|_{L^2}^2 .\end{aligned} \tag{4.38}$$

4.
$$\begin{aligned}\left(\Delta^2 u, F(u)\right)_{L^2} &= \left\| |x|^4 \Delta u \right\|_{L^2}^2 - 16 \left\| |x|^3 \nabla u \right\|_{L^2}^2 - 96 \left\| |x|^2 (x.\nabla u) \right\|_{L^2}^2 \\ &\quad +1512 \left\| |x|^2 u \right\|_{L^2}^2 + a \left\| |x|^4 \nabla \Delta u \right\|_{L^2}^2 - 36a \left\| |x|^3 \Delta u \right\|_{L^2}^2 .\end{aligned} \tag{4.39}$$

5.
$$\begin{aligned}\left(\frac{x}{2}.\nabla \Delta u, F(u)\right)_{L^2} &= \frac{13}{4} \left\| |x|^4 \nabla u \right\|_{L^2}^2 + \frac{11a}{4} \left\| |x|^4 \Delta u \right\|_{L^2}^2 \\ &\quad +4 \left\| |x|^3 (x.\nabla u) \right\|_{L^2}^2 - 180 \left\| |x|^3 u \right\|_{L^2}^2 .\end{aligned} \tag{4.40}$$

Proof: Let us show the first equality. One has

$$(\Delta u, F(u))_{L^2} = \sum_{i=1}^{3} \int_{\mathbb{R}^3} |x|^8 \Delta u_i(x) u_i(x) dx - a \left\| |x|^4 \Delta u \right\|_{L^2}^2 .$$

Integrating twice by parts, we obtain

$$\begin{aligned}
\sum_{i=1}^{3} \int_{\mathbb{R}^3} |x|^8 \Delta u_i(x) u_i(x) dx &= - \left\| |x|^4 \nabla u \right\|_{L^2}^2 - 8 \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} x_j |x|^6 \partial_j u_i(x) u_i(x) dx \\
&= - \left\| |x|^4 \nabla u \right\|_{L^2}^2 - 4 \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} x_j |x|^6 \partial_j \left(u_i^2(x) \right) dx \\
&= - \left\| |x|^4 \nabla u \right\|_{L^2}^2 + 36 \left\| |x|^3 u \right\|_{L^2}^2 ,
\end{aligned} \tag{4.41}$$

which gives the equality (4.36).

We now prove the equality (4.37), integrating by parts, one has

$$\begin{aligned}
\left(\frac{x}{2} . \nabla u, F(u) \right)_{L^2} &= \frac{1}{4} \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} x_j |x|^8 \partial_j \left(u_i^2(x) \right) dx - a \left(\frac{x}{2} . \nabla u, |x|^8 \Delta u \right)_{L^2} \\
&= - \frac{11}{4} \left\| |x|^4 u \right\|_{L^2}^2 - a \left(\frac{x}{2} . \nabla u, |x|^8 \Delta u \right)_{L^2} .
\end{aligned} \tag{4.42}$$

We now have to compute the term of the right hand side of (4.42). Several integrations by parts lead to

$$\begin{aligned}
\left(\frac{x}{2} . \nabla u, |x|^8 \Delta u \right)_{L^2} &= - \frac{1}{4} \sum_{j=1}^{2} \int_{\mathbb{R}^3} |x|^8 x_j \partial_j \left(|\nabla u(x)|^2 \right) dx \\
&\qquad - \frac{1}{2} \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \left(\delta_{j,k} |x|^8 + 8 x_j x_k |x|^6 \right) \partial_j u_i(x) \partial_k u_i(x) dx \\
&= \frac{9}{4} \left\| |x|^4 \nabla u \right\|_{L^2}^2 - 4 \left\| |x|^3 (x . \nabla u) \right\|_{L^2}^2 .
\end{aligned}$$

Thus, going back to (4.42), we get (4.37). The equality (4.38) is now easy to obtain. Indeed, one has

$$
\begin{aligned}
(\mathcal{L}(u), F(u))_{L^2} &= (u, F(u))_{L^2} + (\Delta u, F(u))_{L^2} + \left(\frac{x}{2}.\nabla u, F(u)\right)_{L^2} \\
&= \left\| |x|^4 u \right\|_{L^2}^2 - a\left(u, |x|^8 \Delta u\right)_{L^2} \\
&\quad + 36 \left\| |x|^3 u \right\|_{L^2}^2 - \left\| |x|^4 \nabla u \right\|_{L^2}^2 - a \left\| |x|^4 \Delta u \right\|_{L^2}^2 \\
&\qquad + -\frac{11}{4} \left\| |x|^4 u \right\|_{L^2}^2 - \frac{9a}{4} \left\| |x|^4 \nabla u \right\|_{L^2}^2 + 4a \left\| |x|^3 (x.\nabla u) \right\|_{L^2}^2 \\
&= -\frac{7}{4} \left\| |x|^4 u \right\|_{L^2}^2 - \left(1 + \frac{9a}{4}\right) \left\| |x|^4 \nabla u \right\|_{L^2}^2 - a \left\| |x|^4 \Delta u \right\|_{L^2}^2 \\
&\qquad + 36 \left\| |x|^3 u \right\|_{L^2}^2 + 4a \left\| |x|^3 (x.\nabla u) \right\|_{L^2}^2 - a\left(u, |x|^8 \Delta u\right)_{L^2}.
\end{aligned}
$$

Using the equality (4.41), we get (4.38). The equality (4.39) comes through the same kind of computations. Integrating twice by parts, we have

$$
\begin{aligned}
\left(\Delta^2 u, -|x|^8 a\Delta u\right)_{L^2} &= a \left\| |x|^4 \nabla \Delta u \right\|_{L^2}^2 + 4a \sum_{i=1}^{3} \int_{\mathbb{R}^3} x_j |x|^6 \partial_j \left(|\Delta u(x)|^2\right) dx \\
&= a \left\| |x|^4 \nabla \Delta u \right\|_{L^2}^2 - 36a \left\| |x|^3 \Delta u \right\|_{L^2}^2 .
\end{aligned}
$$

By the same way, we get

$$
\begin{aligned}
\left(\Delta^2 u, |x|^8 u\right)_{L^2} = \left\| |x|^4 \Delta u \right\|_{L^2}^2 &+ 16 \sum_{i=1}^{3} \int_{\mathbb{R}^3} x_i |x|^6 \Delta u(x) \partial_i u(x) dx \\
&+ 72 \int_{\mathbb{R}^3} |x|^6 \Delta u(x) u(x) dx.
\end{aligned}
$$

Integrating again by parts, it comes

$$\begin{aligned}\left(\Delta^2 u, |x|^8 u\right)_{L^2} &= \left\||x|^4 \Delta u\right\|_{L^2}^2 - 8\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} x_i\,|x|^6\,\partial_i\left(|\partial_j u(x)|^2\right)dx - 16\left\||x|^3\nabla u\right\|_{L^2}^2 \\ &\quad - 96\left\||x|^2(x.\nabla u)\right\|_{L^2}^2 - 72\left\||x|^3\nabla u\right\|_{L^2}^2 - 216\sum^{3}\int_{\mathbb{R}^3} x_i\,|x|^4\,\partial_i\left(|u(x)|^2\right)dx \\ &= \left\||x|^4\Delta u\right\|_{L^2}^2 - 16\left\||x|^3\nabla u\right\|_{L^2}^2 - 96\left\||x|^2(x.\nabla u)\right\|_{L^2}^2 + 1512\left\||x|^2 u\right\|_{L^2}^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left(\Delta^2 u, F(u)\right)_{L^2} &= \left\||x|^4\Delta u\right\|_{L^2}^2 - 16\left\||x|^3\nabla u\right\|_{L^2}^2 - 96\left\||x|^2(x.\nabla u)\right\|_{L^2}^2 \\ &\quad + 1512\left\||x|^2 u\right\|_{L^2}^2 + a\left\||x|^4\nabla\Delta u\right\|_{L^2}^2 - 36a\left\||x|^3\Delta u\right\|_{L^2}^2.\end{aligned}$$

It remains to show the equality (4.40) of this lemma, integrating by parts, one has

$$\begin{aligned}\left(\frac{x}{2}.\nabla\Delta u, F(u)\right)_{L^2} &= \left(\frac{x}{2}.\nabla\Delta u, |x|^8 u\right)_{L^2} - a\left(\frac{x}{2}.\nabla\Delta u, |x|^8\Delta u\right)_{L^2} \\ &= \left(\frac{x}{2}.\nabla\Delta u, |x|^8 u\right)_{L^2} + \frac{11a}{4}\left\||x|^4\Delta u\right\|_{L^2}^2.\end{aligned} \tag{4.43}$$

Another integration by parts yields

$$\begin{aligned}\left(\frac{x}{2}.\nabla\Delta u, |x|^8 u\right)_{L^2} &= -\frac{1}{4}\sum_{j}^{3}\int_{\mathbb{R}^3} x_j\,|x|^8\,\partial_j\left(|\nabla u(x)|^2\right)dx \\ &\qquad - \frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k\left(x_j\,|x|^8\right)\partial_j\partial_k u_i(x)u_i(x)dx \\ &= \frac{11}{4}\left\||x|^4\nabla u\right\|_{L^2}^2 \\ &\qquad - \frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\left(\delta_{j,k}\,|x|^8 + 8x_jx_k\,|x|^6\right)\partial_j\partial_k u_i(x)u_i(x)dx. \\ &= \frac{11}{4}\left\||x|^4\nabla u\right\|_{L^2}^2 - \frac{1}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^3}|x|^8\,\Delta u_i(x)u_i(x)dx \\ &\qquad - 4\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} x_jx_k\,|x|^6\,\partial_j\partial_k u_i(x)u_i(x)dx.\end{aligned}$$

The equality (4.41) shows that

$$-\frac{1}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^3}|x|^8\,\Delta u_i(x)u_i(x)dx=\frac{1}{2}\left\||x|^4\,\nabla u\right\|_{L^2}^2-18\left\||x|^3\,u\right\|_{L^2}^2.$$

Thus, we get

$$\begin{aligned}\left(\frac{x}{2}.\nabla\Delta u,|x|^8\,u\right)_{L^2}=\frac{13}{4}\left\||x|^4\,\nabla u\right\|_{L^2}^2-18\left\||x|^3\,u\right\|_{L^2}^2\\ -4\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}x_jx_k\,|x|^6\,\partial_j\partial_k u_i(x)u_i(x)dx.\end{aligned}$$

Integrating one more time by parts, one has

$$\begin{aligned}-4\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}x_jx_k\,|x|^6\,\partial_j\partial_k u_i(x)u_i(x)dx&=4\left\||x|^3\,(x.\nabla u)\right\|_{L^2}^2\\ &\quad+18\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}x_j\,|x|^6\,\partial_j\left(u_i^2(x)\right)dx\\ &=4\left\||x|^3\,(x.\nabla u)\right\|_{L^2}^2-162\left\||x|^3\,u\right\|_{L^2}^2,\end{aligned}$$

and consequently

$$\left(\frac{x}{2}.\nabla\Delta u,|x|^8\,u\right)_{L^2}=\frac{13}{4}\left\||x|^4\,\nabla u\right\|_{L^2}^2+4\left\||x|^3\,(x.\nabla u)\right\|_{L^2}^2-180\left\||x|^3\,u\right\|_{L^2}^2.$$

Going back to (4.43), we get

$$\begin{aligned}&\left(\frac{x}{2}.\nabla\Delta u,F(u)\right)_{L^2}=\\ &\frac{13}{4}\left\||x|^4\,\nabla u\right\|_{L^2}^2+4\left\||x|^3\,(x.\nabla u)\right\|_{L^2}^2-180\left\||x|^3\,u\right\|_{L^2}^2+\frac{11a}{4}\left\||x|^4\,\Delta u\right\|_{L^2}^2.\end{aligned}$$

□

The next lemma enables us to close the $\mathbb{H}^2(4)$ estimate.

Lemma 4.9 *Let* $W \in C^1\left((\tau_0,\tau_\varepsilon),\mathbb{H}^1(4)\right) \cap C^0\left((\tau_0,\tau_\varepsilon),\mathbb{H}^3(4)\right)$ *be the solution of (4.1). There exist two positive constants* γ_0 *and* T_0 *such that, if* $T \geq T_0$ *and* W *satisfy the condition (4.4) for some positive constant such that* $\gamma \leq \gamma_0$*, then there exist* $C > 0$ *such that, for all* $\tau \in [\tau_0, \tau_\varepsilon^*)$*,*

$$\partial_\tau E_5 + 3E_5 + \frac{1}{16}\left\||X|^4 R\right\|_{L^2}^2 + \left(\frac{\alpha}{2}e^{-\tau} + \frac{\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4 \Delta R\right\|_{L^2}^2 \leq K_1 \|R\|_{L^2}^2$$
$$+ CM\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \|\Delta R\|_{L^2}^2\right) + CM\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-4\tau}, \tag{4.44}$$

where K_1 *is a positive constant independent of the parameters.*

Proof: To obtain the inequality 4.44 of this lemma, we perform the L^2−inner product of (4.3) with $|X|^8 (R - \alpha e^{-\tau}\Delta R)$. We deliberately omit the positive terms provided by $\varepsilon\Delta^2 W$ which do not play any role in the next estimates. Using Lemma (4.8) and making some easy computations, one obtains

$$\frac{1}{2}\partial_\tau\left(\left\||X|^4\left(R-\alpha e^{-\tau}\Delta R\right)\right\|_{L^2}^2\right) + \frac{7}{4}\left\||X|^4 R\right\|_{L^2}^2 + \left(1+\frac{7\alpha}{2}e^{-\tau}\right)\left\||X|^4\nabla R\right\|_{L^2}^2$$
$$+\left(\alpha e^{-\tau} + \frac{7\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4 \Delta R\right\|_{L^2}^2 - 108\alpha e^{-\tau}\left\||X|^3 R\right\|_{L^2}^2 =$$
$$36\left\||X|^3 R\right\|_{L^2}^2 + I_1 + I_2 + I_3 + I_4, \tag{4.45}$$

where

$$I_1 = \left(-U.\nabla\left(W-\alpha e^{-\tau}\Delta W\right), |X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},$$

$$I_2 = \left(\left(W-\alpha e^{-\tau}\Delta W\right).\nabla U, |X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},$$

$$I_3 = \left(-\varepsilon e^{-2\tau}\Delta^2\Omega_\infty - \alpha e^{-2\tau}\Delta\Omega_\infty, |X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},$$

$$I_4 = \varepsilon e^{-\tau}\left(16\left\||X|^3\nabla R\right\|_{L^2}^2 + 96\left\||X|^2(X.\nabla R)\right\|_{L^2}^2 + 36\alpha e^{-\tau}\left\||X|^3\Delta R\right\|_{L^2}^2\right).$$

In the proof of this lemma, we use the notation

$$\delta = M\gamma\left(\tfrac{3}{2}-\theta\right)^2.$$

As usual, I_3 is the easiest term to estimate. Indeed, due to the smoothness of Ω_∞ and the inequality (4.6), we get

$$\begin{aligned} I_3 &\leq Ce^{-2\tau} \left\| |X|^4 \left(\alpha\Delta\Omega_\infty + \varepsilon\Delta^2\Omega_\infty\right)\right\|_{L^2} \left(\left\||X|^4 R\right\|_{L^2} + \alpha e^{-\tau} \left\||X|^4 \Delta R\right\|_{L^2}\right) \\ &\leq \mu \left\||X|^4 R\right\|_{L^2}^2 + \mu\alpha^2 e^{-2\tau} \left\||X|^4 \Delta R\right\|_{L^2}^2 + \frac{C\,|b|^2}{\mu} e^{-4\tau} \\ &\leq \mu \left\||X|^4 R\right\|_{L^2}^2 + \mu\alpha^2 e^{-2\tau} \left\||X|^4 \Delta R\right\|_{L^2}^2 + \frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu} e^{-4\tau}, \end{aligned} \tag{4.46}$$

where μ is a positive constant that will be made more precise later.

We now give an estimate of I_4, which is also quite simple to bound. We just need Hölder and Young inequalities to estimate this term in a convenient way. Indeed, using convexity inequalities, it is simple to show that

$$\left\||X|^3 \nabla R\right\|_{L^2}^2 + \left\||X|^2 (X.\nabla R)\right\|_{L^2}^2 \leq C \left\||X|^4 \nabla R\right\|_{L^2}^2 + C \left\|\nabla R\right\|_{L^2}^2,$$

and

$$\alpha e^{-\tau} \left\||X|^3 \Delta R\right\|_{L^2}^2 \leq C\alpha e^{-\tau} \left\||X|^4 \Delta R\right\|_{L^2}^2 + C\alpha e^{-\tau} \left\|\Delta R\right\|_{L^2}^2.$$

Thus, if we take $\varepsilon \leq \alpha M\gamma\left(\frac{3}{2}-\theta\right)^2$, we get

$$\begin{aligned} I_4 \leq CM\gamma\left(\tfrac{3}{2}-\theta\right)^2 &\left(\alpha e^{-\tau} \left\||X|^4 \nabla R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau} \left\||X|^4 \Delta R\right\|_{L^2}^2\right) \\ &+ CM\gamma\left(\tfrac{3}{2}-\theta\right)^2 \left(\alpha e^{-\tau} \left\|\nabla R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau} \left\|\Delta R\right\|_{L^2}^2\right). \end{aligned} \tag{4.47}$$

As for the H^2 estimate, we have to study separately I_1 and I_2. We begin with I_1, that we rewrite

$$I_1 = I_1^1 + I_1^2 + I_1^3,$$

where

$$I_1^1 = \left(U.\nabla\left(R - \alpha e^{-\tau}\Delta R\right), |X|^8 \left(R - \alpha e^{-\tau}\Delta R\right)\right)_{L^2},$$

$$I_1^2 = e^{-2\tau}\left(V_\infty.\nabla\left(\Omega_\infty - \alpha e^{-\tau}\Delta\Omega_\infty\right), |X|^8 \left(R - \alpha e^{-\tau}\Delta R\right)\right)_{L^2},$$

$$I_1^3 = e^{-\tau}\left(K.\nabla\left(\Omega_\infty - \alpha e^{-\tau}\Delta\Omega_\infty\right), |X|^8 \left(R - \alpha e^{-\tau}\Delta R\right)\right)_{L^2}.$$

Using an integration by parts and Hölder inequalities, one has

$$
\begin{aligned}
I_1^1 &= \frac{1}{2}\int_{\mathbb{R}^3} |X|^8\, U(X).\nabla \left(\left|R(X) - \alpha e^{-\tau}\Delta R(X)\right|^2\right) dX \\
&= -4\int_{\mathbb{R}^3} |X|^6\, (X.U(X))\left|R(X) - \alpha e^{-\tau}\Delta R(X)\right|^2 dX \\
&\le C\,\|U\|_{L^\infty}\left(\left\||X|^{7/2}\, R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^{7/2}\,\Delta R\right\|_{L^2}^2\right).
\end{aligned}
$$

The inequalities (2.17) with $p = 2$, $q = 6$ and $\eta = \frac{1}{2}$ and (4.4) and the continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$ give,

$$
\begin{aligned}
I_1^1 &\le C\,\|W\|_{L^2}^{1/2}\,\|W\|_{L^6}^{1/2}\left(\|R\|_{L^2}^2 + \left\||X|^4\, R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\,\|\Delta R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4\,\Delta R\right\|_{L^2}^2\right) \\
&\le C\,\|W\|_{H^1}\left(\|R\|_{L^2}^2 + \left\||X|^4\, R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\,\|\Delta R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4\,\Delta R\right\|_{L^2}^2\right) \\
&\le C M^{1/2}\gamma^{1/2}\left(\frac{3}{2} - \theta\right)\Big(\|R\|_{L^2}^2 + \left\||X|^4\, R\right\|_{L^2}^2 \\
&\qquad\qquad + \alpha^2 e^{-2\tau}\,\|\Delta R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4\,\Delta R\right\|_{L^2}^2\Big).
\end{aligned}
$$

Because of the smoothness of Ω_∞, I_1^2 is a little easier to estimate. Indeed using once more the inequalities (2.17) and (4.6) and Hölder and Young inequalities, we get

$$
\begin{aligned}
I_1^2 &\le Ce^{-2\tau}\,\|V_\infty\|_{L^\infty}\left(\left\||X|^4\,\nabla\Omega_\infty\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4\,\nabla\Delta\Omega_\infty\right\|_{L^2}\right) \\
&\qquad\qquad\qquad\qquad\left(\left\||X|^4\, R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4\,\Delta R\right\|_{L^2}\right) \\
&\le C\,|b|\,e^{-2\tau}\,\|\Omega_\infty\|_{L^2}^{1/2}\,\|\Omega_\infty\|_{L^6}^{1/2}\left(\left\||X|^4\, R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4\,\Delta R\right\|_{L^2}\right) \\
&\le C\,|b|\,e^{-2\tau}\left(\left\||X|^4\, R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4\,\Delta R\right\|_{L^2}\right) \\
&\le \mu\left\||X|^4\, R\right\|_{L^2}^2 + \mu\alpha^2 e^{-2\tau}\left\||X|^4\,\Delta R\right\|_{L^2}^2 + \frac{CM\gamma\left(\frac{3}{2} - \theta\right)^2}{\mu} e^{-4\tau}.
\end{aligned}
$$

Likewise, we get

$$\begin{aligned}
I_1^3 &\le Ce^{-\tau}\left\|K\right\|_{L^\infty}\left(\left\||X|^4 R\right\|_{L^2}+\alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}\right)\\
&\le Ce^{-\tau}\left\|R\right\|_{H^1}\left(\left\||X|^4 R\right\|_{L^2}+\alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}\right)\\
&\le CM^{1/2}\gamma^{1/2}\left(\frac{3}{2}-\theta\right)\left(\left\|R\right\|_{L^2}^2+\left\|\nabla R\right\|_{L^2}^2+\left\||X|^4 R\right\|_{L^2}^2+\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right).
\end{aligned}$$

Finally, taking T so that $\alpha e^{-\tau_0}=\dfrac{\alpha}{T}\le 1$, we have

$$\begin{aligned}
I_1 \le \mu\left\||X|^4 R\right\|_{L^2}^2+\mu\alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau}&\\
+CM^{1/2}\gamma^{1/2}\left(\frac{3}{2}-\theta\right)\Big(\left\|R\right\|_{L^2}^2+\left\|\nabla R\right\|_{L^2}^2+\left\|\Delta R\right\|_{L^2}^2&\\
+\left\||X|^4 R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\Big)&
\end{aligned}\tag{4.48}$$

It remains to bound I_2, which is the hardest term to estimate. Like for I_1, we rewrite it

$$I_2=I_2^1+I_2^2+I_2^3+I_2^4,$$

where

$$\begin{aligned}
I_2^1 &= e^{-\tau}\left(\left(R-\alpha e^{-\tau}\Delta R\right).\nabla V_\infty,|X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},\\
I_2^2 &= \left(\left(R-\alpha e^{-\tau}\Delta R\right).\nabla K,|X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},\\
I_2^3 &= e^{-2\tau}\left(\left(\Omega_\infty-\alpha e^{-\tau}\Delta \Omega_\infty\right).\nabla V_\infty,|X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2},\\
I_2^4 &= e^{-\tau}\left(\left(\Omega_\infty-\alpha e^{-\tau}\Delta \Omega_\infty\right).\nabla K,|X|^8\left(R-\alpha e^{-\tau}\Delta R\right)\right)_{L^2}.
\end{aligned}$$

Using the inequality (2.17) and the smoothness of Ω_∞, we get

$$\begin{aligned}
I_2^1 &\le Ce^{-\tau}\left\|\nabla V_\infty\right\|_{L^\infty}\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le Ce^{-\tau}\left\|\nabla \Omega_\infty\right\|_{L^2}^{1/2}\left\|\nabla \Omega_\infty\right\|_{L^6}^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le C\left|b\right|\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le CM^{1/2}\gamma^{1/2}\left(\frac{3}{2}-\theta\right)\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right).
\end{aligned}$$

We now estimate I_2^2. We recall the notation $\delta = M\gamma\left(\frac{3}{2}-\theta\right)^2$. Using again the inequality (2.17), the inequality (4.6) and the continuous injection of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, one has

$$\begin{aligned}
I_2^2 &\le \|\nabla K\|_{L^\infty}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le C\,\|\nabla R\|_{L_2}^{1/2}\|\nabla R\|_{L_6}^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le C\delta^{1/4}\,\|\nabla R\|_{H^1}^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le C\delta^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\qquad\qquad + C\delta^{1/4}\,\|\Delta R\|_{L^2}^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)\\
&\le C\delta^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right) + C\delta^{1/4}\,\|\Delta R\|_{L^2}^{1/2}\left\||X|^4 R\right\|_{L^2}^2\\
&\qquad\qquad + C\delta^{1/2}\alpha^{7/4}e^{-\frac{7\tau}{4}}\left\||X|^4 \Delta R\right\|_{L^2}^2.
\end{aligned}$$

To finish the estimate of I_2^2, we use the convexity inequality $ab \le \frac{3}{4}a^{\frac{4}{3}} + \frac{1}{4}b^4$ and the condition (4.6). We obtain

$$\begin{aligned}
I_2^2 &\le C\delta^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right) + C\delta^{1/4}\left(\|\Delta R\|_{L^2}^2 + \left\||X|^4 R\right\|_{L^2}^{8/3}\right)\\
&\qquad\qquad + C\delta^{1/2}\alpha^{7/4}e^{-\frac{7\tau}{4}}\left\||X|^4 \Delta R\right\|_{L^2}^2\\
&\le C\delta^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right) + C\delta^{1/4}\,\|\Delta R\|_{L^2}^2 + C\delta^{7/12}\left\||X|^4 R\right\|_{L^2}^2\\
&\qquad\qquad + C\delta^{1/2}\alpha^{7/4}e^{-\frac{7\tau}{4}}\left\||X|^4 \Delta R\right\|_{L^2}^2.
\end{aligned}$$

Consequently, if we assume $\gamma \le 1$ and $\left(\frac{3}{2}-\theta\right) \le 1$, one has

$$I_2^2 \le CM^{7/12}\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}\left(\left\||X|^4 R\right\|_{L^2}^2 + \|\Delta R\|_{L^2}^2 + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right)$$

It it easier bound I_2^3. Indeed, the inequality (2.18) and the inequality (4.6) imply

$$\begin{aligned}
I_2^3 &\le Ce^{-2\tau}\left\|\Omega_\infty - \alpha e^{-\tau}\Delta\Omega_\infty\right\|_{L^\infty}\|\nabla V_\infty\|_{L^2}\left(\left\||X|^4 R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}\right)\\
&\le Ce^{-2\tau}\|\Omega_\infty\|_{L^2}\left(\left\||X|^4 R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}\right)\\
&\le C\,|b|\,e^{-2\tau}\left(\left\||X|^4 R\right\|_{L^2} + \alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}\right)\\
&\le \mu\left\||X|^4 R\right\|_{L^2}^2 + \mu\alpha^2 e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2 + \frac{CM\gamma\left(\frac{3}{2}-1\right)^2}{\mu}e^{-4\tau}.
\end{aligned}$$

Likewise, it comes

$$\begin{aligned}
I_2^4 &\leq Ce^{-\tau}\left\|\Omega_\infty-\alpha e^{-\tau}\Delta\Omega_\infty\right\|_{L^\infty}\|\nabla K\|_{L^2}\left(\left\||X|^4 R\right\|_{L^2}+\alpha e^{-\tau}\left\||X|^4\Delta R\right\|_{L^2}\right)\\
&\leq C\,|b|\,e^{-\tau}\|R\|_{L^2}\left(\left\||X|^4 R\right\|_{L^2}+\alpha e^{-\tau}\left\||X|^4\Delta R\right\|_{L^2}\right)\\
&\leq CM\gamma\left(\frac{3}{2}-1\right)^2\left(\|R\|_{L^2}^2+\left\||X|^4 R\right\|_{L^2}^2+\alpha^2e^{-2\tau}\left\||X|^4\Delta R\right\|_{L^2}^2\right).
\end{aligned}$$

Thus, taking T_0 large enough so that $\alpha e^{-\tau_0}=\frac{\alpha}{T}\leq 1$, the following inequality holds:

$$\begin{aligned}
I_2 \leq{}& \mu\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4\Delta R\right\|_{L^2}^2\right)+\frac{CM\gamma\left(\frac{3}{2}-1\right)^2}{\mu}e^{-4\tau}\\
&+CM\gamma^{1/4}\left(\frac{3}{2}-1\right)^{1/2}\left(\|R\|_{L^2}^2+\|\Delta R\|_{L^2}^2+\left\||X|^4 R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4\Delta R\right\|_{L^2}^2\right).
\end{aligned}\tag{4.49}$$

Combining the equality (4.45) together with the inequalities (4.46), (4.47), (4.48) and (4.49) and taking T_0 big enough compared to α, we have

$$\begin{aligned}
&\frac{1}{2}\partial_\tau\left(\left\||X|^4\left(R-\alpha e^{-\tau}\Delta R\right)\right\|_{L^2}^2\right)+\frac{7}{4}\left\||X|^4 R\right\|_{L^2}^2+\left(1+\frac{7\alpha}{2}e^{-\tau}\right)\left\||X|^4\nabla R\right\|_{L^2}^2\\
&\qquad+\left(\alpha e^{-\tau}+\frac{7\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4\Delta R\right\|_{L^2}^2-108\alpha e^{-\tau}\left\||X|^3 R\right\|_{L^2}^2\leq\\
&C\left(M\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}+\mu\right)\left(\left\||X|^4 R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4\nabla R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4\Delta R\right\|_{L^2}^2\right)\\
&\qquad+36\left\||X|^3 R\right\|_{L^2}^2+CM\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}\left(\|R\|_{L^2}^2+\|\nabla R\|_{L^2}^2+\|\Delta R\|_{L^2}^2\right)\\
&\qquad+\frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau}.
\end{aligned}\tag{4.50}$$

Integrating by parts like in the proof of Lemma 4.8, we have

$$E_4=\frac{1}{2}\left\||X|^4 R\right\|_{L^2}^2+\frac{\alpha^2}{2}e^{-2\tau}\left\||X|^4\Delta R\right\|_{L^2}^2+\alpha e^{-\tau}\left\||X|^4\nabla R\right\|_{L^2}^2-36\alpha e^{-\tau}\left\||X|^3 R\right\|_{L^2}^2.\tag{4.51}$$

Consequently, the inequality (4.50) becomes

$$\begin{aligned}
&\partial_\tau E_5 + 3E_5 + \frac{1}{4}\left\||X|^4 R\right\|_{L^2}^2 + \left(1+\frac{\alpha}{2}e^{-\tau}\right)\left\||X|^4 \nabla R\right\|_{L^2}^2 \\
&\qquad\qquad + \left(\alpha e^{-\tau} + \frac{\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4 \Delta R\right\|_{L^2}^2 \leq \\
&C\left(M\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2} + \mu\right)\left(\left\||X|^4 R\right\|_{L^2}^2 + \alpha e^{-\tau}\left\||X|^4 \nabla R\right\|_{L^2}^2 + \alpha e^{-\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2\right) \\
&\quad +36\left\||X|^3 R\right\|_{L^2}^2 + CM\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \|\Delta R\|_{L^2}^2\right) \\
&\qquad\qquad + \frac{CM\gamma\left(\frac{3}{2}-\theta\right)^2}{\mu}e^{-4\tau}.
\end{aligned} \tag{4.52}$$

Thus, taking γ_0 and μ small enough, we obtain

$$\begin{aligned}
&\partial_\tau E_5 + 3E_5 + \frac{1}{8}\left\||X|^4 R\right\|_{L^2}^2 + \left(\frac{\alpha}{2}e^{-\tau} + \frac{\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4 \Delta R\right\|_{L^2}^2 \leq 36\left\||X|^3 R\right\|_{L^2}^2 \\
&\quad +CM\gamma^{1/4}\left(\frac{3}{2}-\theta\right)^{1/2}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \|\Delta R\|_{L^2}^2\right) + CM\gamma\left(\frac{3}{2}-\theta\right)^2 e^{-4\tau}.
\end{aligned} \tag{4.53}$$

Using Hölder and the convexity inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, a simple computation leads to

$$\left\||X|^3 R\right\|_{L^2}^2 \leq \frac{3\mu^{4/3}}{4}\left\||X|^4 R\right\|_{L^2}^2 + \frac{1}{4\mu^4}\|R\|_{L^2}^2,$$

for all $\mu > 0$.

Using this inequality with μ small enough, we finally obtain

$$\begin{aligned}
&\partial_\tau E_5 + 3E_5 + \frac{1}{16}\left\||X|^4 R\right\|_{L^2}^2 + \left(\frac{\alpha}{2}e^{-\tau} + \frac{\alpha^2}{4}e^{-2\tau}\right)\left\||X|^4 \Delta R\right\|_{L^2}^2 \leq K_1\|R\|_{L^2}^2 \\
&\quad +CM\gamma^{1/4}\left(\tfrac{3}{2}-\theta\right)^{1/2}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \|\Delta R\|_{L^2}^2\right) + CM\gamma\left(\tfrac{3}{2}-\theta\right)^2 e^{-4\tau},
\end{aligned} \tag{4.54}$$

where K_1 is a positive constant.

□

This lemma, combined with the inequality (4.35) enables to finish the $H^2(4)$ estimate of R. We define the functional

$$E_6 = KE_4 + E_5, \tag{4.55}$$

with K some large positive constant that will be made more precise later.

Inequalities (4.35) and (4.44) show that one has

$$\partial_\tau E_6 + 2\theta E_6 + 10K \left\|(-\Delta)^{-\left(\theta-\frac{1}{4}\right)} R\right\|_{L^2}^2 + \frac{K}{2}\|\nabla R\|_{L^2}^2 + \frac{K}{4}\|\Delta R\|_{L^2}^2$$
$$+\frac{1}{16}\left\||X|^4 R\right\|_{L^2}^2 + \frac{\alpha^2}{4}e^{-2\tau}\left\||X|^4 \Delta R\right\|_{L^2}^2 \leq$$
$$K_1\|R\|_{L^2}^2 + CM^2\gamma^{1/4}\left(\|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 + \|\Delta R\|_{L^2}^2 + \left\||X|^4 R\right\|_{L^2}^2\right) + CM^2\gamma e^{-\frac{7\tau}{2}}.$$

Interpolating again $\|R\|_{L^2}^2$ between $\left\|(-\Delta)^{-\left(\theta-\frac{1}{4}\right)} R\right\|_{L^2}^2$ and $\|\nabla R\|_{L^2}^2$ and taking K and γ_0 respectively sufficiently large and small, we get

$$\partial_\tau E_6 + 2\theta E_6 \leq CM^2\gamma e^{-\frac{7\tau}{2}}. \tag{4.56}$$

5 Proof of Theorem 2.1

5.1 Theorem 2.1 for approximate solutions

In this section, under the condition (2.9), we show that the solutions of (4.1) are actually global in time and that the inequality (2.10) of Theorem 2.1 holds for these solutions. To get this result, we take advantage of the energy estimates that we have obtained in the section 4. The following theorem is a copy of Theorem 2.1 for solutions of the regularized system (4.1).

Theorem 5.1 *Let θ be a fixed positive constant such that $0 < \theta < \frac{3}{2}$, ε be a positive constant and $W_0 \in \mathbb{H}^2(4)$. There exist three positive constants $\gamma_0 = \gamma_0(\alpha)$, $\varepsilon = \varepsilon_0(\alpha)$ and $T = T_0(\alpha)$ such that if $T \geq T_0$, $\varepsilon \leq \varepsilon_0$ and there exists a positive constant $\gamma \leq \gamma_0$ such that $W_0 \in \mathbb{H}^2(4)$ satisfies the condition*

$$\|W_0\|_{L^2(4)}^2 + \|\nabla W_0\|_{L^2}^2 + \alpha e^{-\tau_0}\|\Delta W_0\|_{L^2}^2 + \alpha^2 e^{-2\tau_0}\left\||X|^4 \Delta W_0\right\|_{L^2}^2 \leq \gamma\left(\frac{3}{2}-\theta\right)^2, \tag{5.1}$$

where $\tau_0 = \log(T)$,

then there exist a unique solution $W_\varepsilon \in C^1\left((\tau_0, +\infty), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0, +\infty), \mathbb{H}^3(4)\right)$ *to the system (4.1) and a positive constant* $C = C(\alpha, \gamma, \theta, \tau_0)$ *such that, for all* $\tau \geq \tau_0$,

$$\left\| \left(Id - \alpha e^{-\tau}\Delta\right) \left(W_\varepsilon(\tau) - e^{-\tau}\sum_{i=1}^{3} b_i f_i\right) \right\|_{L^2(4)} \leq Ce^{-\theta\tau}, \tag{5.2}$$

where $b_i = \displaystyle\int_{\mathbb{R}^3} p_i(X).W_0(X)dX$.

In order to prove this theorem, we use the energy estimates that we established in the section 4. To obtain the inequality (5.2), we need the energy functional E_6 to be equivalent to the $H^2(4)$-norm of R_ε. If we take K large enough in the definition (4.55) of E_6, then the next lemma holds.

Lemma 5.1 *Let* $R_\varepsilon \in C^1\left((\tau_0, +\infty), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0, +\infty), \mathbb{H}^3(4)\right)$ *and* E_6 *be the energy functional defined by (4.55). There exists* K_0 *such that, if* $K \geq K_0$*, then there exists a positive constant* C *such that*

$$E_6(\tau) \leq C\left(\|R_\varepsilon\|^2_{L^2(4)} + \|\nabla R_\varepsilon\|^2_{L^2} + \alpha e^{-\tau}\|\Delta R_\varepsilon\|^2_{L^2} + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R_\varepsilon\right\|^2_{L^2(4)}\right), \tag{5.3}$$

$$C\left(\|R_\varepsilon\|^2_{L^2(4)} + \|\nabla R_\varepsilon\|^2_{L^2} + \alpha e^{-\tau}\|\Delta R_\varepsilon\|^2_{L^2} + \alpha^2 e^{-2\tau}\left\||X|^4 \Delta R_\varepsilon\right\|^2_{L^2(4)}\right) \leq E_6(\tau). \tag{5.4}$$

Proof: The inequalities (5.3) and (5.4) come directly from the definition of E_6 and the interpolation inequality (4.16).

$\square$

Proof of theorem 5.1: Let θ be a fixed constant such that $0 < \theta < \frac{3}{2}$ and $W_\varepsilon \in C^1\left((\tau_0, +\infty), \mathbb{H}^1(4)\right) \cap C^0\left((\tau_0, +\infty), \mathbb{H}^3(4)\right)$ be the solution of the system (4.1) given by theorem 3.1. Let T and K be sufficiently large so that they satisfy the conditions of the lemmas 4.5, 4.6, 4.7 and 4.9 and assume that the initial data W_0 satisfy the condition (2.9) for some $\gamma > 0$ which will be made more precise later. We decompose W_ε such that

$$W_\varepsilon = e^{-\tau}\Omega_\infty + R_\varepsilon,$$

where $\Omega_\infty = \sum_{i=1}^{3} b_i f_i$, $b_i = \int_{\mathbb{R}^3} p_i(X).W_0(X)dX$ and $\{f_1, f_2, f_3\}$ is the basis of the eigenspace of $\mathcal{L}$ associated to the eigenvalue -1, given by (2.1).

Let M be a positive constant such that $M > 2$ that will be made more precise later and $\tau_\varepsilon^* \in [\tau_0, \tau_\varepsilon]$ be the biggest positive time such that the inequality (4.4) holds. We take γ and ε sufficiently small and T sufficiently large so that the lemmas 4.5, 4.6, 4.7 and 4.9 hold. According to the inequality (4.55), one has, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\partial_\tau \left(E_6(\tau) e^{2\theta\tau} \right) \leq CM^2 \gamma e^{-\left(\frac{7}{2} - 2\theta\right)\tau} \tag{5.5}$$

Integrating in time the previous inequality between τ_0 and $\tau \in [\tau_0, \tau_\varepsilon^*)$, we get

$$E_6(\tau) \leq E_6(\tau_0) e^{-2\theta(\tau - \tau_0)} + CM^2 \gamma e^{-\frac{7\tau_0}{2}} \left(e^{-2\theta(\tau-\tau_0)} - e^{-\frac{7}{2}(\tau - \tau_0)} \right). \tag{5.6}$$

Arguing like in the proof of Lemma 4.1 and using the inequality (5.3), we can show that

$$E_6(\tau_0) \leq C_1 \gamma \left(\frac{3}{2} - \theta \right)^2,$$

which implies

$$E_6(\tau) \leq C\gamma \left(\frac{3}{2} - \theta \right)^2 + CM^2 \gamma e^{-\frac{7\tau_0}{2}}. \tag{5.7}$$

According to the inequalities (5.4) and (4.6), one has, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$|b|^2 + \|R_\varepsilon\|^2_{L^2(4)} + \|\nabla R_\varepsilon\|^2_{L^2} + \alpha e^{-\tau} \|\Delta R_\varepsilon\|^2_{L^2} + \alpha^2 e^{-2\tau} \left\| |X|^4 \Delta R_\varepsilon \right\|^2 \leq C\gamma \left(\frac{3}{2} - \theta \right)^2 + CM^2 \gamma e^{-\frac{7\tau_0}{2}}.$$

Recalling that $W_\varepsilon = \sum_{i=1}^{3} b_i f_i + R_\varepsilon$, we get

$$\|W_\varepsilon\|^2_{L^2(4)} + \|\nabla W_\varepsilon\|^2_{L^2} + \alpha e^{-\tau} \|\Delta W_\varepsilon\|^2_{L^2} + \alpha^2 e^{-2\tau} \left\| |X|^4 \Delta W_\varepsilon \right\|^2 \leq C_1 \gamma \left(\frac{3}{2} - \theta \right)^2 + C_2 M^2 \gamma e^{-\frac{7\tau_0}{2}},$$

where C_1 and C_2 are two positive constants.

We take M sufficiently large so that $C_1 \leq \frac{M}{4}$ and $\tau_0 = \ln(T)$ sufficiently large so that $C_2 M^2 e^{-\frac{7\tau_0}{2}} \leq \dfrac{M\left(\frac{3}{2}-\theta\right)^2}{4}$, we obtain, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\|W_\varepsilon\|_{L^2(4)}^2 + \|\nabla W_\varepsilon\|_{L^2}^2 + \alpha e^{-\tau} \|\Delta W_\varepsilon\|_{L^2}^2 + \alpha^2 e^{-2\tau} \left\||X|^4 \Delta W_\varepsilon\right\|^2 \leq \frac{M\gamma\left(\frac{3}{2}-\theta\right)^2}{2}. \quad (5.8)$$

In particular, the inequality (5.8) shows that $\tau_\varepsilon^* = \tau_\varepsilon$. Furthermore, letting τ tend to τ_ε, we see that if τ_ε is finite, then the $H^1(4)$ norm of W_ε stay bounded on $[\tau, \tau_\varepsilon)$. According to the proof of Theorem 3.1, it implies in particular that one can extend the interval of definition of W_ε over τ_ε. Consequently, we have necessarily $\tau_\varepsilon = +\infty$. In addition, going back to the inequality (5.6) and applying the inequality (5.4) of Lemma 5.1, we see that the inequality (5.2) holds.

□

5.2 Existence of solutions in $\mathbb{H}^2(4)$

In this section, we show that there exists a weak solution to the system (1.5) belonging to the space $C^0([\tau_0, +\infty), \mathbb{H}^2(4))$. To this end, we show that, when ε tends to 0, W_ε tends to a divergence free vector W which satsifies (1.5) in a weak sense. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive terms which tends to 0. Let $W_{\varepsilon_n} \in C^1((\tau_0, +\infty), \mathbb{H}^1(4)) \cap C^0((\tau_0, +\infty), \mathbb{H}^3(4))$ be the global solution of (4.1) given by Theorem 5.1, with initial data W_0. Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^3$. For $s \in \mathbb{R}^+$, $H^s(\mathcal{O})$ denotes the restriction of the Sobolev space $H^s(\mathbb{R}^3)$ on $\mathcal{O}$. For $s \geq 1$, we define also the space

$$H_0^s(\mathcal{O}) = \left\{u \in H^s(\mathcal{O}) : u_{|\partial\mathcal{O}=0}\right\}.$$

Let τ_1 be a fixed positive time such that $\tau_1 > \tau_0$. Due to the boundedness property of W_{ε_n} in $L^\infty([\tau_0, \tau_1], \mathbb{H}^2(4))$ uniformly with respect to n, there exist $W \in L^\infty([\tau_0, \tau_1], \mathbb{H}^2(4))$ and a subsequence of ε_n (that we still note ε_n) such that

$$W_{\varepsilon_n} \rightharpoonup W \quad \text{weak* in} \quad L^\infty\left([\tau_0, \tau_1], H^2(\mathcal{O})^3\right). \quad (5.9)$$

Since W_{ε_n} is bounded in $L^\infty([\tau_0, \tau_1], \mathbb{H}^2(4))$, applying the operator $(Id - \alpha e^{-\tau}\Delta)^{-1}$ to the first equality of (4.1), it is quite easy to see that $\partial_\tau W_{\varepsilon_n}$ is bounded in $L^\infty([\tau_0, T], L^2(\mathcal{O})^3)$

uniformly with respect to n. Consequently, W_{ε_n} is equicontinuous in time on $L^2(\mathcal{O})^3$. Indeed, given σ_1 and σ_2 belonging to $[\tau_0, \tau_1]$, one has

$$\begin{aligned}\|W_{\varepsilon_n}(\sigma_1) - W_{\varepsilon_n}(\sigma_2)\|_{L^2(\mathcal{O})} &= \left\| \int_{\sigma_2}^{\sigma_1} \partial_\tau W_{\varepsilon_n}(s) ds \right\|_{L^2(\mathcal{O})} \\ &\leq \left| \int_{\sigma_2}^{\sigma_1} \|\partial_\tau W_{\varepsilon_n}(s)\|_{L^2(\mathcal{O})} ds \right| \\ &\leq |\sigma_1 - \sigma_2| \max_{s\in[\tau_0,T]} \|\partial_\tau W_{\varepsilon_n}(s)\|_{L^2(\mathcal{O})} .\end{aligned}$$

Besides, for all $\tau \in [\tau_0, \tau_1]$, the set $\bigcup_{n\in\mathbb{N}} W_{\varepsilon_n}(\tau)$ is bounded in $H^2(\mathcal{O})^3$ and thus compact in $L^2(\mathcal{O})^3$. Applying the classical Arzela-Ascoli theorem, we conclude that

$$W_{\varepsilon_n} \longrightarrow W \text{ strongly in } C^0\left([\tau_0, \tau_1], L^2(\mathcal{O})^3\right).$$

A classical interpolation inequality between L^2 and H^2 yields, for all $s < 2$,

$$W_{\varepsilon_n} \longrightarrow W \quad \text{strongly in} \quad C^0\left([\tau_0, \tau_1], H^s(\mathcal{O})^3\right). \tag{5.10}$$

The two identities (5.9) and (5.10) are sufficient to pass to the limit in the weak formulation of the system (4.1) and to show that W is a weak solution of the system (1.5). More precisely, for every $\varphi \in C^1\left([\tau_0, \tau_1], H_0^1(\mathcal{O})^3\right)$ such that $\operatorname{div}\varphi = 0$, one has, for all $\tau \in [\tau_0, \tau_1]$,

$$\begin{aligned}&\int_{\mathcal{O}} \big(W(\tau) - \alpha e^{-\tau}\Delta W(\tau)\big)\, \varphi(\tau) dX + \int_{\tau_0}^{\tau}\int_{\mathcal{O}} \mathcal{L}\Big(W(\sigma)\Big)\varphi(\sigma) dX d\sigma \\ &\qquad + \int_{\tau_0}^{\tau}\int_{\mathcal{O}} U(\sigma) \times \big(W(\sigma) - \alpha e^{-\sigma}\Delta W(\sigma)\big) \operatorname{curl}\varphi(\sigma) dX d\sigma \\ &= \int_{\mathcal{O}} \big(W_0 - \alpha e^{-\tau_0}\Delta W_0\big)\, \varphi(\tau_0) dX + \int_{\tau_0}^{\tau}\int_{\mathcal{O}} \big(W(\sigma) - \alpha e^{-\sigma}\Delta W(\sigma)\big)\, \partial_\tau\varphi(\sigma) dX d\sigma \\ &\qquad + \int_{\tau_0}^{\tau}\int_{\mathcal{O}} \frac{3\alpha}{2} e^{-\sigma}\Delta W(\sigma)\varphi(\sigma) dX d\sigma + \int_{\tau_0}^{\tau}\int_{\mathcal{O}} \frac{\alpha}{2} e^{-\sigma}\Delta W(\sigma)\left(X.\nabla\varphi(\sigma)\right) dX d\sigma.\end{aligned} \tag{5.11}$$

We just show that the non-linear term converges, using (5.9) and (5.10). The other ones are nearly obvious. We have

$$\int_{\tau_0}^{\tau}\int_{\mathcal{O}} U_{\varepsilon_n}(\sigma)\times\left(W_{\varepsilon_n}(\sigma)-\alpha e^{-\sigma}\Delta W_{\varepsilon_n}(\sigma)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma = \int_{\tau_0}^{\tau}\int_{\mathcal{O}} U(\sigma)\times\left(W(\sigma)-\alpha e^{-\sigma}\Delta W(\sigma)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma + R_n + S_n, \tag{5.12}$$

where

$$R_n = \int_{\tau_0}^{\tau}\int_{\mathcal{O}} \left(U(\sigma)-U_{\varepsilon_n}(\sigma)\right)\times\left(W_{\varepsilon_n}(\sigma)-\alpha e^{-\sigma}\Delta W_{\varepsilon_n}(\sigma)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma,$$

$$S_n = \int_{\tau_0}^{\tau}\int_{\mathcal{O}} U(\sigma)\times\left(W(\sigma)-W_{\varepsilon_n}(\sigma)-\alpha e^{-\sigma}\left(\Delta W(\sigma)-\Delta W_{\varepsilon_n}(\sigma)\right)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma.$$

Due to Hölder inequalities, the boundedness property of W_{ε_n} in $H^2(\mathcal{O})^3$ and the inequality (2.17), we have

$$\begin{aligned}
R_n &\leq C\int_{\tau_0}^{\tau}\left\|U(\sigma)-U_{\varepsilon_n}(\sigma)\right\|_{L^\infty(\mathcal{O})}\left\|\nabla\varphi(\sigma)\right\|_{L^2(\mathcal{O})}d\sigma\\
&\leq C\int_{\tau_0}^{\tau}\left\|W(\sigma)-W_{\varepsilon_n}(\sigma)\right\|_{L^2(\mathcal{O})}^{1/2}\left\|W(\sigma)-W_{\varepsilon_n}(\sigma)\right\|_{L^6(\mathcal{O})}^{1/2}\left\|\nabla\varphi(\sigma)\right\|_{L^2(\mathcal{O})}d\sigma\\
&\leq C\left(T-\tau_0\right)\max_{\sigma\in[\tau_0,T]}\left\|W(\sigma)-W_{\varepsilon_n}(\sigma)\right\|_{H^1(\mathcal{O})}\max_{\sigma\in[\tau_0,T]}\left\|\nabla\varphi(\sigma)\right\|_{L^2(\mathcal{O})}.
\end{aligned}$$

Thus, the identity (5.10) implies that $R_n\to 0$ when $n\to+\infty$.

Because of the identity (5.9), it is clear that we have also $S_n\to 0$ when $n\to+\infty$. Thus, we have shown that, for all $\tau\in[\tau_0,\tau_1]$,

$$\lim_{n\to+\infty}\int_{\tau_0}^{\tau}\int_{\mathcal{O}} U_{\varepsilon_n}(\sigma)\times\left(W_{\varepsilon_n}(\sigma)-\alpha e^{-\sigma}\Delta W_{\varepsilon_n}(\sigma)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma = \int_{\tau_0}^{\tau}\int_{\mathcal{O}} U(\sigma)\times\left(W(\sigma)-\alpha e^{-\sigma}\Delta W(\sigma)\right)\operatorname{curl}\varphi(\sigma)dXd\sigma. \tag{5.13}$$

Furthermore, since $W_{\varepsilon_n}(\tau)$ converge weakly to $W(\tau)$ in $\mathbb{H}^2(4)$, from the inequality (5.2), we get

$$\left\|\left(I-\alpha e^{-\tau}\Delta\right)\left(W(\tau)-e^{-\tau}\sum_{i=1}^{3}b_if_i\right)\right\|_{L^2(4)}\leq Ce^{-\theta\tau}, \tag{5.14}$$

for all $\tau \in [\tau_0, +\infty)$.

Uniqueness

It remains to show that the solutions of (1.2) are unique in the space $C^0(\mathbb{R}^+, \mathbb{H}^2(4))$. To show this fact, it suffices to show that the divergence free vector field u obtained from a solution w of (1.2) through the Biot-Savart law is unique. Since w belongs to $C^0(\mathbb{R}^+, \mathbb{H}^2(4))$, the inequality (2.16) with $q = 2$ and $p = \frac{6}{5}$ and the inequality (2.18) with $p = 2$ of the lemma 2.2 imply directly that $u \in C^0(\mathbb{R}^+, H^3(\mathbb{R}^3)^3)$. Furthermore, u satisfies the equations of motion of second grade fluids (1.1). The uniqueness of the H^3-solutions of (1.1) has been shown in [18] for the case of a bounded open set of $\mathbb{R}^3$ with Dirichlet boundary conditions. In our case, we can apply the computations of the proof of [19, Theorem 2], which imply the uniqueness of the solutions of (1.1) with initial data in $H^3(\mathbb{R}^3)^3$.

Chapitre 4

Fluides de grade 3

I. Asymptotic profiles for the third grade fluids equations on $\mathbb{R}^2$

1 Introduction

The study of the behaviour of the non-Newtonian fluids is a significant topic of research in mathematics, but also in physics or biology. Indeed, these fluids, the behaviour of which cannot be described with the classical Navier-Stokes equations, are found everywhere in the nature. For examples, blood, wet sand or certain kind of oils used in industry are non-Newtonian fluids. In this paper, we investigate the behaviour of a particular class of non-Newtonian fluids that is the third grade fluids, which are a particular case to the Rivlin-Ericksen fluids (see [59], [62]). The constitutive law of such fluids is defined through the Rivlin-Ericksen tensors, given recursively by

$$\begin{aligned} &A_1 = \nabla u + (\nabla u)^t\,, \\ &A_k = \partial_t A_{k-1} + u.\nabla A_{k-1} + (\nabla u)^t A_{k-1} + A_{k-1}\nabla u, \end{aligned}$$

where u is a divergence free vector field of $\mathbb{R}^2$ or $\mathbb{R}^3$ which represents the velocity of the fluid. The most famous example of a Rivlin-Ericksen fluid is the class of the Newtonian fluids, which are modelized through the stress tensor

$$\sigma = -pId + \nu A_1,$$

where $\nu > 0$ is the kinematic viscosity and p is the pressure which depends on u. Introduced into the equations of conservation of momentum, this stress tensor leads to the well known Navier-Stokes equations.

In this article, we consider a larger class of fluids, for which the stress tensor is not linear in the Rivlin-Ericksen tensors, but a polynomial function of degree 3. As introduced by Fosdick and Rajagopal in [31], the stress tensor that we consider is defined by

$$\sigma = -pId + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta \left|A_1\right|^2 A_1,$$

where $\nu > 0$ is the kinematic viscosity, p is the pressure, $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ and $\beta \geq 0$.

We assume in this article that the density of the fluid is constant in space and time and equals to 1. Actually, the value of the density is not significant, since we can replace the parameters ν, α_1, α_2 and β by dividing them by the density. Introduced into the equations of conservation of momentum, the tensor σ leads to the system

$$\begin{aligned}
&\partial_t \left(u - \alpha_1 \Delta u\right) - \nu \Delta u + \text{curl }\left(u - \alpha_1 \Delta u\right) \wedge u \\
&\qquad\qquad\qquad - \left(\alpha_1 + \alpha_2\right)\left(A.\Delta u + 2\text{div }\left(LL^t\right)\right) - \beta \text{div }\left(\left|A\right|^2 A\right) + \nabla p = 0, \\
&\text{div } u = 0, \\
&u_{|t=0} = u_0,
\end{aligned} \tag{1.1}$$

where $L = \nabla u$, $A(u) = \nabla u + \left(\nabla u\right)^t$ and $\wedge$ denotes the classical vectorial product on $\mathbb{R}^3$. For matrices $A, B \in \mathcal{M}_d(\mathbb{R})$, we define $A : B = \displaystyle\sum_{i,j=1}^{d} A_{i,j} B_{i,j}$ and $\left|A\right|^2 = A : A$. If the space dimension is 2, we use the convention $u = (u_1, u_2, 0)$ and curl $u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$. Notice also that if $\alpha_1 + \alpha_2 = 0$ and $\beta = 0$, we recover the equations of fluids of grade 2, which is another class of non-Newtonian fluids, introduced earlier by Dunn and Fosdick in 1974 (see [24], [35] or [19]). If in addition $\alpha_1 = 0$, then one recovers the classical Navier-Stokes equations.

The system of equations (1.1) has been studied in various cases, on bounded domains of $\mathbb{R}^2$ or $\mathbb{R}^3$ or in the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$ (see [1], [5], [9], [10], [11] and [55]). On a bounded domain Ω of $\mathbb{R}^d$, $d = 2, 3$, with Dirichlet boundary conditions, Amrouche and Cioranescu have shown the existence of local solutions to (1.1) when the initial data belong to the Sobolev space $H^3(\Omega)^d$ (see [1]). In addition, these solutions are unique. For this study, the authors have assumed the restriction

$$\left|\alpha_1 + \alpha_2\right| \leq \left(24\nu\beta\right)^{1/2},$$

which is justified by thermodynamics considerations. The proof of their result is obtained using a Galerkin method with functions belonging to the eigenspaces of the operator curl $(Id - \alpha_1 \Delta)$. In dimension 3, a slightly different method has been applied by D. Bresch and J. Lemoine, who used Schauder's fixed point Theorem to extend the result of [1] to the case of initial data belonging to the Sobolev space $W^{2,r}(\Omega)^3$. They have shown in [9] the local existence of unique solutions of (1.1) in the space $C^0\left([0,T], W^{2,r}(\Omega)^3\right)$, where $T > 0$. In addition, if the data are small enough in the space $W^{2,r}(\Omega)^3$, the solutions are global in time. Notice also that the existence of such solutions holds without restrictions on the parameters of the system (1.1).

In the case of third grade fluids filling the whole space $\mathbb{R}^d$, $d = 2, 3$, Busuioc and Iftimie established the existence of global solutions with initial data belonging to $H^2(\mathbb{R}^d)^d$, without restrictions on the parameters or on the size of the data (see [10]). In this study, the authors used a Friedrichs method and performed a priori estimates in H^2 which allow to show the existence of solutions of (1.1) in the space $L^\infty_{loc}\left(\mathbb{R}^+, H^2(\mathbb{R}^d)^d\right)$. Besides, these solutions are unique if $d = 2$. Later, Paicu extended the results of [10] to the case of initial data belonging to $H^1(\mathbb{R}^d)^d$, assuming additional restrictions on the parameters of the equation ; the uniqueness is not known in this space (see [55]). The method that he used is slightly different from the one used in [10]. Indeed, although Paicu also considered a Friedrichs scheme, the convergence of the approximate solutions to a solution of (1.1) is done via a monotonicity method. Notice that Theorem 1.1 of this article shows the existence of solutions of the equations of third grade fluids on $\mathbb{R}^2$ for initial data in weighted Sobolev spaces (see Section 3).

In what follows, we consider a third grade fluid filling the whole space $\mathbb{R}^2$. Actually, the equations that we consider are not exactly the system (1.1) but the one satisfied by $w = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$. In dimension 2, the vorticity equations of the third garde fluids are given by

$$\begin{aligned} &\partial_t (w - \alpha_1 \Delta w) - \nu \Delta w + u.\nabla (w - \alpha_1 \Delta w) \\ &\qquad\qquad -\beta \operatorname{div}\left(|A|^2 \nabla w\right) - \beta \operatorname{div}\left(\nabla\left(|A|^2\right) \wedge A\right) = 0, \\ &\operatorname{div} u = 0, \\ &w_{|t=0} = w_0 = \operatorname{curl} u_0. \end{aligned} \tag{1.2}$$

Notice that the parameter α_2 does no longer appear in (1.2) and thus does not play any role in the study of these equations. Indeed, due to the divergence free property of u, a short computation shows that curl $(A.\Delta u + 2\operatorname{div}(LL^t)) = 0$, or equivalently there exists q such that $A.\Delta u + 2\operatorname{div}(LL^t) = \nabla q$. This phenomenon is very particular to the dimension 2 and does not occur in dimension 3. Notice also that the previous system is autonomous in w. Indeed, the vector field u depends on w and can be recovered from w via the Biot-Savart law, which is a way to get a divergence free vector field such that $\operatorname{curl} u = w$. The motivation for considering the vorticity equations instead of the equations of motion comes from the fact that, due to spectral reasons, we have to study the behaviour of the solutions of (1.2) in weighted Lebesgue spaces. Unfortunately, these functions spaces are not suitable for the equations of motions and are not preserved by the system (1.1).

In this article, we establish the existence and uniqueness of solutions of (1.2) in weighted Sobolev spaces, but the main aim is the study of the asymptotic behaviour of these solutions when t goes to infinity. More precisely, we want to describe the first order asymptotic profiles of the solutions of (1.2). We consider a fluid of third grade which fills $\mathbb{R}^2$ without forcing term applied to it. In this case, as it is expected, the solutions of (1.2) tend to 0 as t goes to infinity. Our motivation is to show that these solutions behave like those of the Navier-Stokes equations. In our case, we will show that the solutions of (1.2) behave asymptotically like solutions of the heat equations, up to a constant that we can compute from the initial data. The methods that we use in the present paper are based on scaled variables and energy estimates in several functions spaces. This work is inspired by several older results obtained for other fluid mechanics equations. The first and second order asymptotic profiles have been described for the Navier-Stokes equations in dimensions 2 and 3 by Gallay and Wayne (see[36], [37], [38] and [39]). In dimension 2, they have shown in [36] and [38] that the first order asymptotic profiles of Navier-Stokes equations are given up to a constant by a smooth Gaussian function called the Oseen vortex sheet. More precisely, for a solution w of the vorticity Navier-Stokes equations (that is the system 1.2 with $\alpha_1 = \beta = 0$), for every $2 \leq p \leq +\infty$, the following property holds:

$$\left\| w(t) - \frac{\int_{\mathbb{R}^2} w_0(x)dx}{t} G\left(\frac{\cdot}{\sqrt{t}}\right) \right\|_{L^p} = \mathcal{O}(t^{-\frac{3}{2}+\frac{1}{p}}), \text{ when } t \to +\infty,$$

where G is the Oseen vortex sheet

$$G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}. \tag{1.3}$$

The methods that they used in [36] are very different from the ones that we develop in this article. Although they also considered scaled variables, the convergence to the asymptotic profiles is not obtained through energy estimates. Indeed, using dynamical systems arguments, they established the existence of a finite-dimensional manifold which is locally invariant by the semiflow associated to the Navier-Stokes equations. Then, they showed that, under restrictions on the size of the data, the solutions of the Navier-Stokes equations behave asymptotically like solutions on this invariant manifold. The description of the asymptotic profiles is thus obtained by the description of the dynamics of the Navier-Stokes equations on the invariant manifold. Later, the smallness assumption on the data has been removed (see [38]). In [47], Jaffal-Mourtada describes the first order asymptotics of second grade fluids, under smallness assumptions on the initial data in weighted Sobolev spaces. She has shown that the solutions of the second grade fluids equations converge also to the Oseen vortex sheet. In this paper, we apply the methods used by Jaffal-Mourtada, namely scaled variables and energy estimates. According to these results, one can say that the fluids of second grade behave asymptotically like Newtonian fluids. In this paper, we show that, under the same smallness assumptions on the initial data, the same behaviour occurs for third grade fluids equations. We emphasize that the rate of convergence that we obtain is better than the one obtained in [47]. Actually, we show that we can choose the rate of convergence as close as wanted to the optimal one, assuming that the initial data are small enough. Since second grade fluids are a particular case of third grade fluids, we establish an improvement of the rate obtained in [47]. Actually, the main difference between third and second grade fluids equations in dimension 2 is the presence of the additional term $\beta \mathrm{div}\left(|A|^2 A\right)$ in the third grade fluids equations. Sometimes, this term helps to obtain global estimates, like in [10] or [55], but introduces additional difficulties when one looks for estimates in H^3 or in more regular Sobolev spaces (see [1], [5] or [11]). Here, we have to establish estimates in weighted Sobolev spaces with H^2 regularity for the vorticity w, which is harder than doing estimates in H^3 for u.

We next introduce scaled variables. In order to simplify the notations, we assume that $\nu = 1$. Let $T > 1$ be a positive constant which is introduced in order to avoid restrictions on the size of the parameter α_1 and which will be made more precise later. We consider the solution w of (1.2) and define W and U such that $\mathrm{curl}\, U = W$ through the change of variables $X = \dfrac{x}{\sqrt{t+T}}$ and $\tau = \log(t+T)$. We set

$$\begin{cases} u(t,x) = \dfrac{1}{\sqrt{t+T}} U\left(\log(t+T), \dfrac{x}{\sqrt{t+T}}\right), \\ w(t,x) = \dfrac{1}{t+T} W\left(\log(t+T), \dfrac{x}{\sqrt{t+T}}\right). \end{cases} \tag{1.4}$$

For $\tau \geq \log(T)$, we have

$$\begin{cases} U(\tau, X) = e^{\tau/2} u\left(e^{\tau} - T, e^{\tau/2} X\right), \\ W(\tau, X) = e^{\tau} w\left(e^{\tau} - T, e^{\tau/2} X\right). \end{cases} \tag{1.5}$$

These variables, called scaled or self-similar variables have been introduced in order to study the long time asymptotic of solutions of parabolic equations and particularly to show the convergence to self-similar solutions (see [25], [26], [33] or [48]), that is to say under the form $\dfrac{1}{t+T}F\left(\dfrac{x}{\sqrt{t+T}}\right)$.

Scaled variables have been used to deal with the asymptotic behaviour of many equations, not necessarily parabolic ones (see [13], [14] [47], [40] or [41]). For instance, in [40], Gallay and Raugel described the first and second order asymptotic profiles in weighted Sobolev spaces for damped wave equations, using scaled variables. In [41], they use scaled variables to show a stability result of hyperbolic fronts for the same equations.

For sake of simplicity, we set $A^{i,j} = \partial_j U_i + \partial_i U_j$. Considering self-similar variables, one can see that W and its corresponding divergence free vector field U satisfy the system

$$\begin{aligned}
&\partial_\tau \left(W - \alpha_1 e^{-\tau}\Delta W\right) - \mathcal{L}(W) + U.\nabla\left(W - \alpha_1 e^{-\tau}\Delta W\right) + \alpha_1 e^{-\tau}\Delta W\\
&\qquad\qquad +\alpha_1 e^{-\tau}\tfrac{X}{2}.\nabla\Delta W - \beta e^{-2\tau}\mathrm{div}\,\left(|A|^2\,\nabla W\right) - \beta e^{-2\tau}\mathrm{div}\,\left(\nabla\left(|A|^2\right)\wedge A\right) = 0,\\
&\mathrm{div}\, U = 0,\\
&W_{|\tau=\tau_0} = W_0,
\end{aligned} \tag{1.6}$$

where $\tau_0 = \log(T)$, $W_0(X) = e^{\tau_0} w_0\left(e^{\tau_0/2}X\right)$ and $\mathcal{L}$ is the linear differential operator defined by

$$\mathcal{L}(W) = \Delta W + W + \tfrac{X}{2}.\nabla W.$$

Notice that the initial time of the system (1.6) is $\log(T)$. By choosing T sufficiently large, one can consider $\alpha_1 e^{-\tau}$ as small as wanted. This fact will allow to study the behaviour of the solutions of (1.6) without restrictions on the size of α_1. Formally, we see that most of the terms of the system (1.6) tend to 0 as time goes to infinity. We show that the solutions of (1.6) asymptotically behave like solutions of

$$\partial_\tau W_\infty = \mathcal{L}(W_\infty). \tag{1.7}$$

In order to describe the solutions of the system (1.7), we have to study the spectrum of the linear differential operator $\mathcal{L}$ in appropriate functions spaces. The form of the previous system and the definition of $\mathcal{L}$ lead to consider weighted Lebesgue spaces. For $m \in \mathbb{N}$, we define

$$L^2(m) = \left\{u \in L^2(\mathbb{R}^2) : \left(1 + |x|^2\right)^{m/2} u \in L^2(\mathbb{R}^2)\right\},$$

equipped with the norm

$$\|u\|_{L^2(m)} = \left(\int_{\mathbb{R}^2} \left(1 + |x|^2\right)^m |u(x)|^2 \, dx \right)^{1/2}.$$

The spectrum of $\mathcal{L}$ in $L^2(m)$ is studied in details in [36, Appendix A]. It is composed of the discrete spectrum

$$\sigma_d(\mathcal{L}) = \left\{ -\tfrac{k}{2} : k \in \{0, 1, ..., m-2\} \right\},$$

and the continuous spectrum

$$\sigma_c(\mathcal{L}) = \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\tfrac{m-1}{2} \right\}.$$

In particular, the eigenvalue 0 is simple and the Oseen vortex G given by (1.3) is an eigenfunction of $\mathcal{L}$ associated to 0. Of course, G is a solution of (1.7) and we will show that the solutions of (1.6) behave like G when the time goes to infinity. To this end, we decompose the solutions W of (1.6) as follows

$$W(\tau) = \eta G + f(\tau),$$

where $\eta \in \mathbb{R}$ will be made more precise later and $f(\tau)$ is a rest which will tend to 0 as τ goes to infinity.

In order to get a good rate of convergence for f, we shall "push" the continuous spectrum of $\mathcal{L}$ to the left by choosing an appropriate weighted Lebesgue space. For this reason, we work in $L^2(2)$, so that $\sigma_c(\mathcal{L}) = \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\frac{1}{2} \right\}$. Since the second eigenvalue of $\mathcal{L}$ in $L^2(2)$ is $-\frac{1}{2}$, the best result that we expect is as follows

$$f(\tau) = \mathcal{O}(e^{-\tau/2}) \quad \text{in } L^2(2), \quad \text{when} \quad \tau \to +\infty.$$

Notice that choosing a weighted space $L^2(m)$ with $m > 2$ would be useless for describing the first order asymptotics only. Indeed, if we take $m > 2$, the second eigenvalue would still be $-\frac{1}{2}$ and the rate of convergence could not be better than $e^{-\tau/2}$.

For later use, we define the divergence free vector field V such that $\operatorname{curl} V = G$. It is obtained by the Biot-Savart law and given by

$$V(X) = \frac{1 - e^{-\frac{|X|^2}{4}}}{2\pi |X|^2} \begin{pmatrix} -X_2 \\ X_1 \end{pmatrix}.$$

In particular, for every $X \in \mathbb{R}^2$, one has

$$V(X).X = 0, \;\; V(X).\nabla G(X) = 0 \;\text{ and }\; V(X).\nabla\Delta G(X) = 0.$$

Before stating the main theorem of this paper, we have to define some additional functions spaces. For $m \in \mathbb{N}$, we set

$$H^1(m) = \left\{u \in L^2(m) : \partial_j u \in L^2(m); j \in \{1,2\}\right\},$$
$$H^2(m) = \left\{u \in H^1(m) : \partial_j u \in H^1(m); j \in \{1,2\}\right\},$$

equipped with the norms

$$\|u\|_{H^1(m)} = \left(\|u\|^2_{L^2(m)} + \|\nabla u\|^2_{L^2(m)}\right)^{1/2} \quad \text{and}$$
$$\|u\|_{H^2(m)} = \left(\|u\|^2_{H^1(m)} + \|\nabla^2 u\|^2_{L^2(m)}\right)^{1/2},$$

where $|\nabla u|^2 = \sum_{i=1}^{2} (\partial_i u)^2$ and $\left|\nabla^2 u\right|^2 = \sum_{i,j=1}^{2} (\partial_i \partial_j u)^2$.

The following theorem describes the first order asymptotic profile of W in $H^2(2)$, if one assumes that the initial data W_0 are small enough in the weighted Sobolev space $H^2(2)$.

Theorem 1.1 *Let θ be a constant such that $0 < \theta < 1$. There exist two positive constants $\gamma_0 = \gamma_0(\alpha_1, \beta)$ and $T_0 = T_0(\alpha_1) \geq 1$ such that, for all $W_0 \in H^2(2)$ satisfying the condition*

$$\|W_0\|^2_{H^1} + \frac{\alpha_1}{T}\|\Delta W_0\|^2_{L^2} + \left\||X|^2 W_0\right\|^2_{L^2} + \frac{\alpha_1^2}{T^2}\left\||X|^2 \Delta W_0\right\|^2_{L^2} \leq \gamma(1-\theta)^6, \tag{1.8}$$

for some $T \geq T_0$ and $0 < \gamma \leq \gamma_0$,

there exist a unique global solution $W \in C^0\left([\tau_0, +\infty), H^2(2)\right)$ of (1.6) and a positive constant $C = C(\alpha_1, \beta, \theta)$ such that, for all $\tau \geq \tau_0$,

$$\left\|\left(1 - \alpha_1 e^{-\tau}\Delta\right)(W(\tau) - \eta G)\right\|^2_{L^2(2)} \leq C\gamma e^{-\theta\tau}, \tag{1.9}$$

where $\eta = \int_{\mathbb{R}^2} W_0(X)dX$, $\tau_0 = \log(T)$ and the parameters α_1 and β are fixed and given in (1.1).

Remark 1.1 *The smallness assumption (1.8) is not optimal. By working harder, it is possible to get $\gamma(1-\theta)^p$ with $p < 6$ in the right hand side of the inequality.*

Remark 1.2 *Notice that Theorem 1.1 establishes an improvement of [47, Theorem 1.1] concerning the first order asymptotics of the second grade fluids equations. Indeed, the above theorem holds also with $\beta = 0$ and consequently describes the first order asymptotic profiles of the solutions of the second grade fluids equation. The improvement comes from the fact that one can choose θ as close as wanted to 1, which is the optimal rate. In [47], the constant θ can not be bigger than $\frac{1}{2}$.*

Theorem 1.1 implies the following result in the unscaled variables.

Corollary 1.1 *Let θ be a constant such that $0 < \theta < 1$. There exist two positive constants $\gamma_0 = \gamma_0(\alpha_1, \beta)$ and $T_0 = T_0(\alpha_1, \beta) \geq 1$ such that, for all $w_0 \in H^2(2)$ satisfying the condition*

$$T \|w_0\|_{L^2}^2 + T^2 \|\nabla w_0\|_{L^2}^2 + \frac{1}{T} \left\| |x|^2 w_0 \right\|_{L^2}^2 + \alpha_1 T^3 \|\Delta w_0\|_{L^2}^2 + \frac{\alpha_1^2}{T} \left\| |x|^2 \Delta w_0 \right\|_{L^2}^2 \leq \gamma (1-\theta)^6 , \tag{1.10}$$

for some $T \geq T_0$ and $0 < \gamma \leq \gamma_0$,

there exists a unique global solution $w \in C^0([0, +\infty), H^2(2))$ of (1.2) such that, for all $1 \leq p \leq 2$, there exists a positive constant $C = C(\alpha_1, \beta, \theta)$ such that, for all $t \geq 0$,

$$\|(1 - \alpha_1 \Delta)(w(t) - \eta \Omega(t))\|_{L^p} \leq C\gamma (t+T)^{-1+\frac{1}{p}-\frac{\theta}{2}} ,$$

and there exists a positive constant $C = C(\alpha_1, \beta, \theta)$ such that, for all $t \geq 0$,

$$\left\| |x|^2 (1 - \alpha_1 \Delta)(w(t) - \eta \Omega(t)) \right\|_{L^2} \leq C\gamma (t+T)^{\frac{1}{2}-\frac{\theta}{2}} ,$$

where $\eta = \int_{\mathbb{R}^2} w_0(x) dx$ and $\Omega(t, x) = \frac{1}{t+T} G\left(\frac{x}{\sqrt{t+T}}\right)$.

Theorem 1.1 describes the asymptotic behaviour of the solutions of (1.6) in $H^2(2)$ at the first order. Since the solutions of Navier-Stokes equations converge also to the Oseen vortex sheet, we can say that the fluids of third grade behave asymptotically like Newtonian fluids. Notice that the functions space $H^2(2)$ is suitable for the first order asymptotics because it "pushes" the continuous spectrum of $\mathcal{L}$ far enough to get 0 as an isolated eigenvalue. If we had to describe the asymptotics of (1.6) at the second order, we should work in a space where $\mathcal{L}$ has at least two isolated eigenvalue. Due to the forms of σ_c and σ_d, the second order asymptotics must be studied in functions space with polynomial weight of degree at least 3, in order to get the two isolated eigenvalues 0 and $-\frac{1}{2}$.

Notice also that as the system (1.2) and our change of variables preserve the total mass, we have, for all $\tau \geq \tau_0$ and $t \geq 0$,

$$\eta = \int_{\mathbb{R}^2} w_0(x)dx = \int_{\mathbb{R}^2} w(t,x)dx = \int_{\mathbb{R}^2} W_0(X)dX = \int_{\mathbb{R}^2} W(\tau,X)dX.$$

The plan of this article is as follows. In Section 2, we recall classical results concerning the Biot-Savart law and give several technical lemmas. In Section 3, we introduce a regularized system, which is close to (1.6) and depends on a small parameter $\varepsilon > 0$. Actually, we add the regularizing term $\varepsilon\Delta^2 W$ to the system (1.6) and show the existence of unique regular solutions W_ε to this new system. In Section 4, using energy estimates in various functions spaces, we show that W_ε satisfies the inequality (1.9) of Theorem 1.1, and thus tends to the Oseen vortex sheet G when τ goes to infinity. In Section 5, we let ε go to 0 and show that W_ε tends in a sense to a solution W of (1.6). Additionally, this solution satisfies the inequality (1.9) of Theorem 1.1 and consequently tends also to the Oseen vortex sheet. Finally, we establish the uniqueness of W, which enables us to say that every solution of (1.6) satisfying the assumption (1.8) converges to the Oseen Vortex sheet when τ goes to infinity.

2 Biot-Savart law and auxiliary lemmas

In this section, we state several technical lemmas which are useful to prove Theorem 1.1. These lemmas concern the Biot-Savart and inequalities involving weighted Lebesgue norms. The first one is needed when making energy estimates in weighted Sobolev spaces. In what follows, we use the notation

$$\|u\| = \|u\|_{L^2},$$

and C denotes a positive constant which can depend on the fixed constants α_1 and β.

The first lemma will be useful in Section 4 to obtain estimates in Sobolev spaces of negative order. We define, for $s \in \mathbb{R}$, the operator $(-\Delta)^s$, given by

$$(-\Delta)^s u = \bar{\mathcal{F}}\left(|\xi|^{2s}\,\widehat{u}\right),$$

where $\widehat{u}$ (also denoted $\mathcal{F}(u)$) is the Fourier transform of u, given by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^2} u(x)e^{-ix.\xi}dx,$$

and $\bar{\mathcal{F}}$ denotes the inverse Fourier transform

$$\bar{\mathcal{F}}(v)(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} v(\xi) e^{ix.\xi} d\xi.$$

Lemma 2.1 *Let s be a positive real number such that $\frac{3}{4} < s < 1$, then we have the following two inequalities.*

1. *Let $g \in L^2(1)$. Then $(-\Delta)^{-s} \nabla g \in L^2(\mathbb{R}^2)$ and there exists $C > 0$ independent of g and s such that*

$$\left\| (-\Delta)^{-s} \nabla g \right\| \leq \frac{C}{(1-s)^{3/2}} \|g\|_{L^2(1)} . \tag{2.1}$$

2. *Let $g \in L^2(2)$ such that $\displaystyle\int_{\mathbb{R}^2} g(x)dx = 0$. Then $(-\Delta)^{-s} g \in L^2(\mathbb{R}^2)$ and there exists $C > 0$ independent of g and s such that*

$$\left\| (-\Delta)^{-s} g \right\|_{L^2} \leq \frac{C}{(1-s)^{3/2}} \|g\|_{L^2(2)} . \tag{2.2}$$

Proof : We start by proving the inequality (2.1). For $j \in \{1, 2\}$, using Fourier variables, one has

$$\begin{aligned}
\left\| (-\Delta)^{-s} \partial_j g \right\|_{L^2}^2 &\leq C \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} |\widehat{g}|^2 \, d\xi + \|g\|_{L^2}^2 \\
&\leq C \left(\int_{|\xi|\leq 1} \frac{1}{|\xi|^{2s}} d\xi \right)^{\frac{2s-1}{s}} \left(\int_{|\xi|\leq 1} |\widehat{g}|^{\frac{2s}{1-s}} \, d\xi \right)^{\frac{1-s}{s}} + \|g\|_{L^2}^2 \\
&\leq \frac{C}{(1-s)} \|\widehat{g}\|_{L^{\frac{2s}{1-s}}}^2 + \|g\|_{L^2}^2 .
\end{aligned}$$

We now use the continuous injection of $H^1(\mathbb{R}^2)$ into $L^{\frac{2s}{1-s}}(\mathbb{R}^2)$. Looking at the computations of [15, p. 723-724], one can see that there exists a constant $C > 0$ such that

$$\|u\|_{L^p} \leq Cp \|u\|_{H^1} , \text{ for all } u \in H^1(\mathbb{R}^2) \text{ and } 2 \leq p < +\infty. \tag{2.3}$$

Notice that Cp is not the optimal constant in the previous inequality. Using the inequality (2.3), one has

$$\begin{aligned}
\left\| (-\Delta)^{-s} \partial_j g \right\|_{L^2}^2 &\leq \frac{C}{(1-s)^3} \|\widehat{g}\|_{H^1}^2 + \|g\|_{L^2}^2 \\
&\leq \frac{C}{(1-s)^3} \|g\|_{L^2(1)}^2 .
\end{aligned}$$

We now prove the inequality (2.2). Since $\int_{\mathbb{R}^2} f(x)dx = 0$, using Fourier variables, we get

$$\begin{aligned}
\left\|(-\Delta)^{-s} g\right\|_{L^2}^2 &= (2\pi)^2 \int_{\mathbb{R}^2} \frac{1}{|\xi|^{4s}} |\widehat{g}(\xi)|^2 \, d\xi \\
&\leq (2\pi)^2 \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s}} |\widehat{g}(\xi)|^2 \, d\xi + \|g\|_{L^2}^2 \\
&\leq (2\pi)^2 \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s}} \left| \int_0^1 \xi . \nabla \widehat{g}(\sigma\xi) d\sigma \right|^2 d\xi + \|g\|_{L^2}^2 \\
&\leq C \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} \left| \int_0^1 |\nabla \widehat{g}(\sigma\xi)| \, d\sigma \right|^2 d\xi + \|g\|_{L^2}^2 .
\end{aligned}$$

Cauchy-Schwarz inequality and Fubini's theorem give

$$\left\|(-\Delta)^{-s} g\right\|_{L^2}^2 \leq C \int_0^1 \int_{|\xi|\leq 1} \frac{1}{|\xi|^{4s-2}} |\nabla \widehat{g}(\sigma\xi)|^2 \, d\xi d\sigma + \|g\|_{L^2}^2 .$$

Using Hölder inequality, we get

$$\begin{aligned}
\left\|(-\Delta)^{-s} g\right\|_{L^2}^2 &\leq C \int_0^1 \left(\int_{|\xi|\leq 1} \frac{1}{|\xi|^{4-\frac{2}{s}}} d\xi \right)^s \left(\int_{|\xi|\leq 1} |\nabla \widehat{g}(\sigma\xi)|^{\frac{2}{1-s}} \, d\xi \right)^{1-s} d\sigma + \|g\|_{L^2}^2 \\
&\leq C \left(\frac{s}{1-s} \right)^s \int_0^1 \left(\int_{|\xi|\leq 1} |\nabla \widehat{g}(\sigma\xi)|^{\frac{2}{1-s}} \, d\xi \right)^{1-s} d\sigma + \|g\|_{L^2}^2 .
\end{aligned}$$

The change of variables $\zeta = \sigma\xi$ yield

$$\begin{aligned}
\left\|(-\Delta)^{-s} g\right\|_{L^2}^2 &\leq C \left(\frac{s}{1-s} \right)^s \int_0^1 \left(\int_{|\zeta|\leq \sigma} \frac{1}{\sigma^2} |\nabla \widehat{g}(\zeta)|^{\frac{2}{1-s}} \, d\zeta \right)^{1-s} d\sigma + \|g\|_{L^2}^2 \\
&\leq C \left(\frac{s}{1-s} \right)^s \left(\frac{1}{2s-1} \right) \|\nabla \widehat{g}\|_{L^{\frac{2}{1-s}}}^2 + \|g\|_{L^2}^2 .
\end{aligned}$$

Finally, we use again the inequality (2.3) and obtain

$$\begin{aligned}
\left\|(-\Delta)^{-s} g\right\|_{L^2}^2 &\leq C \left(\frac{s}{1-s} \right)^s \left(\frac{1}{2s-1} \right) \left(\frac{2}{1-s} \right)^2 \|\widehat{g}\|_{H^2}^2 + \|g\|_{L^2}^2 \\
&\leq \frac{C}{(1-s)^3} \|g\|_{L^2(2)}^2 ,
\end{aligned}$$

which concludes the proof of this lemma.

$\square$

Lemma 2.2 *1. Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^2)$ such that $|x|^2 f \in L^p(\mathbb{R}^2)$, then $|x| f \in L^p(\mathbb{R}^2)$ and the following inequality holds :*

$$\||x| f\|_{L^p} \leq \|f\|_{L^p}^{1/2} \left\||x|^2 f\right\|_{L^p}^{1/2}. \tag{2.4}$$

2. Let $f \in H^2(2)$, there exists $C > 0$ such that

$$\left\||x|^2 \nabla^2 f\right\| \leq C \left(\|f\| + \||x| \nabla f\| + \left\||x|^2 \Delta f\right\|\right). \tag{2.5}$$

3. Let $f \in H^2(2)$, then $|x|^2 \nabla f \in L^4(\mathbb{R}^2)$ and there exists $C > 0$ such that

$$\left\||x|^2 \nabla f\right\|_{L^4} \leq C \left\||x|^2 \nabla f\right\|^{1/2} \left(\|f\|^{1/2} + \||x| \nabla f\|^{1/2} + \left\||x|^2 \Delta f\right\|^{1/2}\right). \tag{2.6}$$

Proof: The inequality (2.4) comes directly from Hölder's inequality. To prove the inequality (2.5), we show by a simple calculation that, for every $j, k \in \{1, 2\}$,

$$\left\||x|^2 \partial_j \partial_k f\right\|^2 \leq C \left(\|f\|^2 + \||x| \nabla f\|^2 + \left\||x|^2 \Delta f\right\|^2\right). \tag{2.7}$$

Indeed, we notice that

$$|x|^2 \partial_j \partial_k f = \partial_j \partial_k \left(|x|^2 f\right) - 2\delta_{j,k} f - 2x_j \partial_k f - 2x_k \partial_j f, \tag{2.8}$$

and furthermore

$$\left\|\partial_j \partial_k \left(|x|^2 f\right)\right\|^2 \leq C \left\|\Delta \left(|x|^2 f\right)\right\|^2 \leq C \left(\|f\|^2 + \||x| \nabla f\|^2 + \left\||x|^2 \Delta f\right\|^2\right). \tag{2.9}$$

Combining (2.8) and (2.9) we get the inequality (2.7).

To obtain (2.6), we use Gagliardo-Niremberg's inequality as follows:

$$\begin{aligned} \left\||x|^2 \nabla f\right\|_{L^4} &\leq C \left\||x|^2 \nabla f\right\|^{1/2} \left\|\nabla \left(|x|^2 \nabla f\right)\right\|^{1/2} \\ &\leq C \left\||x|^2 \nabla f\right\|^{1/2} \left(\||x| \nabla f\|^{1/2} + \left\||x|^2 \nabla^2 f\right\|^{1/2}\right), \end{aligned}$$

and consequently inequality (2.5) implies (2.6).

$\square$

Biot-Savart law: Let w be a real function defined on $\mathbb{R}^2$. The Bio-Savart law is a way to build a divergence free vector field u such that curl $u = w$. It is given by

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} w(y)dy, \tag{2.10}$$

where $(x_1, x_2)^{\perp} = (-x_2, x_1)$.

The next two lemmas give estimates on the divergence free vector field u obtained from w via the Bio-Savart law.

Lemma 2.3 *Let u be the divergence free vector field given by (2.10).*

1. *Assume that $1 < p < 2 < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. If $w \in L^p(\mathbb{R}^2)$, then $u \in L^q(\mathbb{R}^2)^2$ and there exists $C > 0$ such that*

$$\|u\|_{L^q} \leq C \|w\|_{L^p} . \tag{2.11}$$

2. *Assume that $1 \leq p < 2 < q \leq \infty$, and define $\alpha \in (0,1)$ by the relation $\frac{1}{2} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. If $w \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, then $u \in L^\infty(\mathbb{R}^2)^2$ and there exists $C > 0$ such that*

$$\|u\|_{L^\infty} \leq C \|w\|_{L^p}^{\alpha} \|w\|_{L^q}^{1-\alpha} . \tag{2.12}$$

3. *Assume that $1 < p < \infty$. If $w \in L^p(\mathbb{R}^2)$, then $\nabla u \in L^p(\mathbb{R}^2)^4$ and there exists $C > 0$ such that*

$$\|\nabla u\|_{L^p} \leq C \|w\|_{L^p} . \tag{2.13}$$

In addition, div $u = 0$ and curl $u = w$.

We refer to [36] for the proof of this lemma.

Lemma 2.4 *Let u be the divergence free vector field given by (2.10).*

1. *If $w \in L^2(2)$, then $u \in L^4(\mathbb{R}^2)^2$ and there exists $C > 0$ such that*

$$\|u\|_{L^4} \leq C \|w\|_{L^2(2)} . \tag{2.14}$$

2. *If $w \in L^2(2) \cap H^1(\mathbb{R}^2)$, then $u \in L^\infty(\mathbb{R}^2)^2$ and there exists $C > 0$ such that*

$$\|u\|_{L^\infty} \leq C \|w\|_{H^1}^{1/2} \|w\|_{L^2(2)}^{1/2} . \tag{2.15}$$

3. *Let $s \in \mathbb{R}$. If $(-\Delta)^{\frac{s-1}{2}} w \in L^2(\mathbb{R}^2)$ for $s \in \mathbb{R}$, then $(-\Delta)^{s/2} u \in L^2(\mathbb{R}^2)^2$ and there exists $C > 0$ such that*

$$\left\| (-\Delta)^{s/2} u \right\| \leq C \left\| (-\Delta)^{\frac{s-1}{2}} w \right\|. \tag{2.16}$$

4. *Let $s \in \mathbb{R}$. If $w \in H^s(\mathbb{R}^2)$, then $\nabla u \in H^s(\mathbb{R}^2)^4$ and there exists $C > 0$ such that*

$$\|\nabla u\|_{H^s} \leq C \|w\|_{H^s}. \tag{2.17}$$

The proof of the two first inequalities are shown in [47]. The two other inequalities are obvious when using Fourier variables. The next lemma is useful to get energy estimates in weighted Sobolev spaces for solutions of (1.6). For a vector field u, we set

$$|\nabla^2 u|^2 = \sum_{i,j,k=1}^{2} (\partial_j \partial_k u_i)^2 \text{ and } \quad |\nabla^3 u|^2 = \sum_{i,j,k,l=1}^{2} (\partial_j \partial_k \partial_l u_i)^2.$$

Lemma 2.5 *Let $w \in L^2(\mathbb{R}^2)$ and u be the divergence free vector given by (2.10).*

1. *If $w \in H^1(1)$, then $\nabla^2 u \in L^2(1)$ and there exists $C > 0$ such that*

$$\left\|\nabla^2 u\right\|_{L^2(1)} \leq C \left(\|w\|_{H^1} + \||x| \nabla w\| \right). \tag{2.18}$$

2. *If $w \in H^2(1)$, then $|x| \nabla^2 u \in L^4(\mathbb{R}^2)$ and there exists $C > 0$ such that*

$$\left\||x| \nabla^2 u\right\|_{L^4} \leq C \left(\|w\| + \||x| \nabla w\| \right)^{1/2} \left(\|\nabla w\| + \||x| \Delta w\| \right)^{1/2}. \tag{2.19}$$

3. *If $w \in H^2(2)$, then $u \in |x|^2 \nabla^3 u \in L^2(\mathbb{R}^2)$ and there exists $C > 0$ such that*

$$\left\||x|^2 \nabla^3 u\right\| \leq C \left(\|w\| + \||x| \nabla w\| + \left\||x|^2 \Delta w\right\| \right). \tag{2.20}$$

4. *If $w \in L^2(1)$ and $\displaystyle\int_{\mathbb{R}^2} w(x)dx = 0$, then $u \in H^1(1)$ and there exists a positive constant C such that*

$$\|u\| + \||x| \nabla u\| \leq C \||x| w\|. \tag{2.21}$$

5. *If $w \in H^1(2)$ and $\displaystyle\int_{\mathbb{R}^2} w(x)dx = 0$, then $|x|^2 \nabla^2 u \in L^2(\mathbb{R}^2)$ and there exists a positive constant C such that*

$$\left\||x|^2 \nabla^2 u\right\| \leq C \|w\|_{H^1(2)}. \tag{2.22}$$

Proof: Let us show the inequality (2.18). Let w belong to $H^1(1)$ and u be the divergence free vector field obtained via the Biot-Savart law. From the inequality 2.13 of Lemma 2.3, we obtain

$$\left\|\nabla^2 u\right\|_{L^2} \leq C \left\|\nabla w\right\|_{L^2}. \tag{2.23}$$

Since the divergence of u vanishes and since we are in dimension 2, it is enough to show the inequality

$$\left\|x_i \partial_j^2 u_k\right\| \leq C \left(\|w\| + \||x| \nabla w\|\right), \tag{2.24}$$

where $i, j, k \in \{1, 2\}$.

We omit k that doesn't appear in the following calculations. One has

$$\begin{aligned}
\left\||x_i| \partial_j^2 u\right\|^2 &= (2\pi)^2 \int_{\mathbb{R}^2} \left|\partial_i \left(\xi_j^2 \widehat{u}\right)\right|^2 d\xi \\
&\leq C \int_{\mathbb{R}^2} \left|\xi_j \widehat{u}\right|^2 d\xi + C \int_{\mathbb{R}^2} \left|\xi_j^2 \partial_i \widehat{u}\right|^2 d\xi \\
&\leq C \|\nabla u\|^2 + C \int_{\mathbb{R}^2} \left|\mathcal{F}\left(\Delta\left(x_i u\right)\right)\right|^2 d\xi \\
&\leq C \|\nabla u\|^2 + C \||x| \Delta u\|^2.
\end{aligned}$$

Using the inequality (2.13) of Lemma 2.3 with $p = 2$ and remarking that $\partial_1 w = \Delta u_2$ and $\partial_2 w = \Delta u_1$, we obtain (2.24). Combining it with the inequality (2.23), we get (2.18).

The inequality (2.19) is a direct consequence of (2.18) and Gagliardo-Niremberg inequality. Indeed, one has

$$\begin{aligned}
\left\|x_i \partial_j^2 u\right\|_{L^4} &\leq C \left\|x_i \partial_j^2 u\right\|^{1/2} \left\|\nabla \left(x_i \partial_j^2 u\right)\right\|^{1/2} \\
&\leq C \left\|x_i \partial_j^2 u\right\|^{1/2} \left(\left\|\partial_j^2 u\right\| + \left\|x_i \partial_j^2 \nabla u\right\|\right)^{1/2}.
\end{aligned}$$

Furthermore, the inequalities (2.13) and (2.18) yield

$$\left\|x_i \partial_j^2 u\right\|_{L^4} \quad \leq C \left(\|w\| + \||x| \nabla w\|\right)^{1/2} \left(\|\nabla w\| + \|x_i \nabla^2 w\|\right)^{1/2}.$$

Making the same computations than the ones we made to establish (2.18), we obtain

$$\|x_i \nabla^2 w\| \leq C \left(\|\nabla w\| + \||x| \Delta w\|\right),$$

which gives

$$\left\| x_i \partial_j^2 u \right\|_{L^4} \leq C \left(\|w\| + \||x| \nabla w\| \right)^{1/2} \left(\|\nabla w\| + \||x| \Delta w\| \right)^{1/2},$$

and the inequality (2.19) comes when summing for $i \in \{1, 2\}$.

In order to get the inequality (2.20), it suffices to obtain it for $|x|^2 \partial_j \partial_k^2 u$, where $j, k \in \{1, 2\}$. One has

$$\begin{aligned}
\left\| |x|^2 \partial_j \partial_k^2 u \right\|^2 &= (2\pi)^2 \int_{\mathbb{R}^2} \left| \Delta \left(\xi_j \xi_k^2 \widehat{u} \right) \right|^2 d\xi \\
&\leq C \left(\int_{\mathbb{R}^2} \left| |\xi|^2 (\xi_j + \xi_k) \Delta \widehat{u} \right|^2 d\xi + \int_{\mathbb{R}^2} \left| (\xi_j + \xi_k) \widehat{u} \right|^2 d\xi + \int_{\mathbb{R}^2} \left| |\xi|^2 \nabla \widehat{u} \right|^2 d\xi \right) \\
&\leq C \left(\left\| \nabla \Delta \left(|x|^2 u \right) \right\|^2 + \|\nabla u\|^2 + \sum_{i=1}^{2} \|\Delta (x_i u)\|^2 \right) \\
&\leq C \left(\left\| |x|^2 \nabla \Delta u \right\|^2 + \|\nabla u\|^2 + \||x| \Delta u\|^2 \right) \\
&\leq C \left(\left\| |x|^2 \nabla^2 w \right\|^2 + \|w\|^2 + \||x| \nabla w\|^2 \right).
\end{aligned}$$

Applying the inequality (2.5), we get (2.20). The proof of the inequality (2.21) is made in two steps. It is shown in [47] that

$$\|u\| \leq C \||x| w\|. \tag{2.25}$$

To finish the proof of the inequality (2.21), we notice that

$$\||x| w\|^2 = \||x| \partial_1 u_2\|^2 + \||x| \partial_2 u_1\|^2 - 2 \int_{\mathbb{R}^2} |x|^2 \partial_1 u_2 \partial_2 u_1 dx. \tag{2.26}$$

Integrating by parts, one gets

$$\begin{aligned}
-2 \int_{\mathbb{R}^2} |x|^2 \partial_1 u_2 \partial_2 u_1 dx = &\int_{\mathbb{R}^2} |x|^2 u_2 \partial_1 \partial_2 u_1 dx + 2 \int_{\mathbb{R}^2} x_1 u_2 \partial_2 u_1 dx \\
&+ \int_{\mathbb{R}^2} |x|^2 \partial_1 \partial_2 u_2 u_1 dx + 2 \int_{\mathbb{R}^2} x_2 \partial_1 u_2 u_1 dx.
\end{aligned}$$

Using the divergence free property of u and integrating by parts, we have

$$-2\int_{\mathbb{R}^2} |x|^2 \partial_1 u_2 \partial_2 u_1 dx = \||x|\, \partial_1 u_1\|^2 + \||x|\, \partial_2 u_2\|^2 + 4\int_{\mathbb{R}^2} x_2 u_2 \partial_2 u_2 dx + 4\int_{\mathbb{R}^2} x_1 \partial_1 u_1 u_1 dx.$$

Finally, integrating again by parts, we get

$$-2\int_{\mathbb{R}^2} |x|^2 \partial_1 u_2 \partial_2 u_1 dx = \||x|\, \partial_1 u_1\|^2 + \||x|\, \partial_2 u_2\|^2 - 2\, \|u\|^2 .$$

Thus, going back to (2.26), one has

$$\||x|\, \nabla u\|^2 = \||x|\, w\|^2 + 2\, \|u\|^2 .$$

Combining this equality with (2.25), we get inequality (2.21). The inequality (2.22) is obtained in the same way.

□

3 Approximate solutions

In this section, we introduce a "regularized" system of equations, whose solutions are more regular than the solutions of (1.2). Actually, this new system is very close to (1.2), and is obtained by adding the small term $\varepsilon\Delta^2 w$ to (1.2). Here, the positive constant ε is supposed to be small and is devoted to tend to 0. Adding this term, we are able to prove the existence of solutions to the regularized system via a semi-group method. The presence of the term $u.\nabla\Delta w$ would not let us obtain solutions to (1.2) by a semi-group method because of the too high degree of derivatives in this term compared to the linear term Δw. We introduce now the following regularized system of equations:

$$\begin{aligned} &\partial_t \left(w_\varepsilon - \alpha_1 \Delta w_\varepsilon\right) + \varepsilon\Delta^2 w_\varepsilon - \Delta w_\varepsilon + u_\varepsilon \nabla \left(w_\varepsilon - \alpha_1 \Delta w_\varepsilon\right) \\ &\qquad\qquad -\beta \mathrm{div}\, \left(|A_\varepsilon|^2 \nabla w_\varepsilon\right) - \beta \mathrm{div}\, \left(\nabla\left(|A_\varepsilon|^2\right) \wedge A_\varepsilon\right) = 0, \\ &w_{\varepsilon|t=0} = w_0 \in H^2(2), \end{aligned} \tag{3.1}$$

where $A_\varepsilon = \nabla u_\varepsilon + (\nabla u_\varepsilon)^t$.

The aim of this section is to prove the following theorem.

Theorem 3.1 *Let $w_0 \in H^2(2)$. For all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ and a unique solution w_ε of the system* (3.1) *such that*

$$w_\varepsilon \in C^1\left((0,t_\varepsilon)\,, H^1(2)\right) \cap C^0\left([0,t_\varepsilon)\,, H^2(2)\right) \cap C^0\left((0,t_\varepsilon)\,, H^3(2)\right).$$

Proof: First of all, we introduce the change of variable $\tilde{x} = \gamma x$, where γ is a positive constant that is close to 0 and will be made more precise later. This is made in order to not have to consider restrictions on the size of α_1. We note $v_\varepsilon(x) = w_\varepsilon(x/\gamma)$. The system (3.1) provides a new system in v_ε, that we will solve in $H^2(2)$.

$$\begin{aligned} &\partial_t \left(v_\varepsilon - \alpha_1\gamma^2\Delta v_\varepsilon\right) + \varepsilon\gamma^4\Delta^2 v_\varepsilon - \gamma^2\Delta v_\varepsilon + \gamma u_\varepsilon.\nabla\left(v_\varepsilon - \alpha_1\gamma^2\Delta v_\varepsilon\right) \\ &\qquad\qquad -\beta\gamma\nabla\left(|A_\varepsilon|^2\right).\nabla v_\varepsilon - \beta\gamma^2|A_\varepsilon|^2\Delta v_\varepsilon - \beta\text{div}\left(\nabla\left(|A_\varepsilon|^2\right)\wedge A_\varepsilon\right) = 0, \\ &v_{\varepsilon|t=0} = w_0(x/\gamma) \in H^2(2). \end{aligned} \tag{3.2}$$

Although there are terms involving u_ε in this system, it is actually autonomous. In fact, one recover w_ε from v_ε and then recover u_ε via the Biot-Savart law (2.10) applied to w_ε. We set

$$z_\varepsilon(x) = q(x)v_\varepsilon(x),$$

where $q(x) = \left(1 + |x|^2\right)$.

To show the existence of a solution in $H^2(2)$ to the system (3.2), we are reduced to show that there exists a solution in $H^2(\mathbb{R}^2)$ of the system

$$\begin{aligned} &\partial_t\left(z_\varepsilon - \gamma^2\alpha_1\Delta z_\varepsilon - \alpha_1\gamma^2 q\Delta q^{-1}z_\varepsilon - 2\gamma^2\alpha_1 q\nabla q^{-1}.\nabla z_\varepsilon\right) + \varepsilon\gamma^4\Delta^2 z_\varepsilon = F\left(z_\varepsilon\right), \\ &z_{\varepsilon|t=0} = qw_0(x/\gamma) \in H^2(\mathbb{R}^2), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} F(z_\varepsilon) = &-\varepsilon\gamma^4 q\Delta^2\left(q^{-1}z_\varepsilon\right) + \gamma^2 q\Delta\left(q^{-1}z_\varepsilon\right) - \gamma q u_\varepsilon\nabla\left(q^{-1}z_\varepsilon - \gamma^2\alpha_1\Delta\left(q^{-1}z_\varepsilon\right)\right) \\ &+\beta\gamma q\nabla\left(|A_\varepsilon|^2\right).\nabla\left(q^{-1}z_\varepsilon\right) + \beta\gamma^2 q|A_\varepsilon|^2\Delta\left(q^{-1}z_\varepsilon\right) + \beta q\text{div}\left(\nabla\left(|A_\varepsilon|^2\right)\wedge A_\varepsilon\right). \end{aligned} \tag{3.4}$$

We define the two linear operators $B : D(B) = H^1(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2)$ and $D : D(D) = L^2(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2)$ as follows:

$$B(z) = \alpha_1\gamma^2\Delta z + \alpha_1\gamma^2 q\Delta q^{-1}z,$$

$$D(z) = 2\alpha_1\gamma^2 q\nabla q^{-1}.\nabla z.$$

Via Lax-Milgram theorem, it is easy to show that $A = (I - B - D)$ is invertible. We define the bilinear form on $H^1(\mathbb{R}^2)$

$$a(u,v) = (u,v)_{L^2} + \alpha_1\gamma^2\left(\nabla u,\nabla v\right)_{L^2} - \alpha_1\gamma^2\left(q\Delta q^{-1}u,v\right)_{L^2} - 2\alpha_1\gamma^2\left(q\nabla q^{-1}.\nabla u,v\right)_{L^2}.$$

We notice that a is obviously coninuous on $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Using the fact that $q\Delta q^{-1}$ and $q\nabla q^{-1}$ are bounded on $\mathbb{R}^2$, one has, for all $u, v \in H^1(\mathbb{R}^2)$,

$$|a(u,v)| \leq C(\alpha_1, \gamma) \|u\|_{H^1} \|v\|_{H^1},$$

where $C(\alpha_1, \gamma)$ is a positive constant depending on α_1 and γ.

We show now that a is coercive. Via an integration by parts, we get

$$a(u,u) = \|u\|^2 + \alpha_1\gamma^2 \|\nabla u\|^2 - \alpha_1\gamma^2 \int_{\mathbb{R}^2} q\Delta q^{-1} |u|^2 \, dx + \alpha_1\gamma^2 \int_{\mathbb{R}^2} \mathrm{div}\, \left(q\nabla q^{-1}\right) |u|^2 \, dx.$$

Due to the boundedness of $q\Delta q^{-1}$ and div $(q\nabla q^{-1})$, there exists $C > 0$ such that

$$a(u,u) \geq \left(1 - \alpha_1\gamma^2 C\right) \|u\|^2 + \alpha_1\gamma^2 \|\nabla u\|^2.$$

If we take γ sufficiently small, the bilinear form a is both continuous and coercive on $H^1(\mathbb{R}^2)$. From the Lax-Milgram theorem, we conclude that for all $f \in H^{-1}(\mathbb{R}^2)$ there exists $u \in H^1(\mathbb{R}^2)$ such that

$$a(u,v) = \langle f, v\rangle_{H^{-1}\times H^1} \quad \text{for all} \quad v \in H^1(\mathbb{R}^2), \tag{3.5}$$

and consequently $(I - B - D)^{-1}$ is defined from $H^{-1}(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$. We define A : $D(A) = H^3(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$ the linear differential operator on $H^1(\mathbb{R}^2)$

$$A = \varepsilon\gamma^4 \left(I - B - D\right)^{-1} \Delta^2.$$

We rewrite the system (3.3) as follows:

$$\begin{aligned} &\partial_t z_\varepsilon + A\left(z_\varepsilon\right) = \widetilde{F}\left(z_\varepsilon\right), \\ &z_{\varepsilon|t=0} = q w_0(x/\gamma) \in H^2(\mathbb{R}^2), \end{aligned} \tag{3.6}$$

where $\widetilde{F}(z_\varepsilon) = (I - B - D)^{-1} F(z_\varepsilon)$.

To finish the proof of this theorem, we show that the operator A is sectorial on $H^1(\mathbb{R}^2)$, which is equivalent to the fact that $-A$ generates an analytic semigroup on $H^1(\mathbb{R}^2)$. Then, we check that $\widetilde{F}$ is locally Lipschitz from bounded sets of a Sobolev space $H^s(\mathbb{R}^2)$

to $H^1(\mathbb{R}^2)$, where $1 \leq s < 3$. By a theorem that one can found in [58], we get theorem 5.1. A small computation leads to

$$\begin{aligned} A &= \varepsilon\gamma^4 (I-B)^{-1} \Delta^2 - \varepsilon\gamma^4 (I-B-D)^{-1} D (I-B)^{-1} \Delta^2 \\ &= I + \varepsilon\gamma^4 (I-B)^{-1} \Delta^2 - I - \varepsilon\gamma^4 (I-B-D)^{-1} D (I-B)^{-1} \Delta^2 \\ &= J + R, \end{aligned}$$

where

$$\begin{aligned} J &= I + \varepsilon\gamma^4 (I-B)^{-1} \Delta^2, \\ R &= -I - \varepsilon\gamma^4 (I-B-D)^{-1} D (I-B)^{-1} \Delta^2. \end{aligned}$$

Using the same method as the one used to invert $(I-B-D)$, one can invert $(I-B)$ and define $(I-B)^{-1}$ from $H^{-1}(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$. Consequently, J is well defined from $H^3(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$. In the remaining of this proof, we will show that $-J$ generates an analytic semi-group on $H^1(\mathbb{R}^2)$ and then show that R satisfies the conditions of [58, Theorem 2.1 p. 81]. According to this result, it implies that $-A$ generates an analytic semi-group on $H^1(\mathbb{R}^2)$. In order to show that J is sectorial on $H^1(\mathbb{R}^2)$, we associate it to a continuous and coercive bilinear form on $H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$. To this end, we define a H^1-scalar product which is suitable to J. Let us define, for $u, v \in H^1(\mathbb{R}^2)$, the bilinear form on H^1 given by

$$\langle u, v\rangle_{H^1} = \left(\left(1 - \alpha_1\gamma^2 q\Delta q^{-1}\right) u, v\right)_{L^2} + \alpha_1\gamma^2 \left(\nabla u, \nabla v\right)_{L^2}.$$

If γ is sufficiently small compared to α_1, then $\langle .,.\rangle_{H^1}$ is a scalar product on $H^1(\mathbb{R}^2)$. Furthermore, for $u \in H^2(\mathbb{R}^2)$ and $v \in H^1(\mathbb{R}^2)$, one has

$$\langle u, v\rangle_{H^1} = \left((I-B) u, v\right)_{L^2}.$$

We define, using this scalar product, the bilinear form j on $H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ associated to J by the formula

$$j(u,v) = \langle u, v\rangle_{H^1} + \varepsilon\gamma^4 \left(\Delta u, \Delta v\right)_{L^2}.$$

A short computation shows that, for $u \in H^3(\mathbb{R}^2)$ and $v \in H^2(\mathbb{R}^2)$, one has

$$j(u,v) = \langle Ju, v\rangle_{H^1}. \tag{3.7}$$

Furthermore, if γ is small enough, using the definition of $\langle .,.\rangle_{H^1}$ and j, we see that there exists $C(\alpha_1, \varepsilon, \gamma) > 0$ such that, for all $u, v \in H^2(\mathbb{R}^2)$,

$$j(u,v) \leq C(\alpha_1, \varepsilon, \gamma) \left\|u\right\|_{H^2} \left\|v\right\|_{H^2}.$$

Besides, it is simple to check that, if γ is mall enough, there exists $C(\alpha_1, \gamma, \varepsilon) > 0$ such that, for all $u \in H^2(\mathbb{R}^2)$,

$$j(u,u) \geq C(\alpha_1, \gamma, \varepsilon) \|u\|_{H^2}^2 .$$

The bilinear form j is thus coninuous and coercive on $H^2(\mathbb{R}^2)$ and the operator J is sectorial on $H^1(\mathbb{R}^2)$. Additionally, The linear operator R is defined from $H^2(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$, and one can check that there exists $C(\alpha_1, \gamma, \varepsilon) > 0$ such that, for all $u \in H^3(\mathbb{R}^2)$,

$$\|Ru\|_{H^1} \leq C(\alpha_1, \gamma, \varepsilon) \|u\|_{H^2} . \tag{3.8}$$

Applying the equality (3.7) to $u \in H^3(\mathbb{R}^2)$, we get

$$j(u,u) = \langle Ju, u \rangle_{H^1} , \text{ for all } u \in H^3(\mathbb{R}^2).$$

Because j is coercive on H^2, we obtain, via Cauchy-Schwartz inequality,

$$\|u\|_{H^2}^2 \leq C(\alpha_1, \gamma, \varepsilon) \|Ju\|_{H^1} \|u\|_{H^1} , \text{ for all } u \in H^3(\mathbb{R}^2).$$

Going back to (3.8), the following property holds

$$\|Ru\|_{H^1}^2 \leq C(\alpha_1, \gamma, \varepsilon) \|Ju\|_{H^1} \|u\|_{H^1} , \text{ for all } u \in H^3(\mathbb{R}^2).$$

In particular, the Young inequality yields, for all $\delta > 0$,

$$\|Ru\|_{H^1}^2 \leq \delta \|Ju\|_{H^1}^2 + C(\alpha_1, \gamma, \varepsilon) \|u\|_{H^1}^2 , \text{ for all } u \in H^3(\mathbb{R}^2).$$

By a classical result that we can find in [46], $-A$ is thus the generator of an analytic semigroup on $H^1(\mathbb{R}^2)$.

Lastly, it is easy to check that $\widetilde{F}$ is Lipschitzian from the bounded sets of $H^2(\mathbb{R}^2)$ into $H^1(\mathbb{R}^2)$. Combining several results from [46, chapter 3] and [58, section 6.3], we conclude that there exists $t_\varepsilon > 0$ and a unique solution $z_\varepsilon \in C^1\left((0,t_\varepsilon), H^1(\mathbb{R}^2)\right) \cap C^0\left([0,t_\varepsilon), H^2(\mathbb{R}^2)\right) \cap C^0\left((0,t_\varepsilon), H^3(\mathbb{R}^2)\right)$ of the system (3.3). Thus, there exists a unique solution $w_\varepsilon \in C^1\left((0,t_\varepsilon), H^1(2)\right) \cap C^0\left([0,t_\varepsilon), H^2(2)\right) \cap C^0\left((0,t_\varepsilon), H^3(2)\right)$ to the system (3.1).

$\square$

4 Energy estimates

In this section, we perform energy estimates on the regularized solutions of the third grade fluids equations in the weighted space $H^2(2)$. These estimates are independent of ε and will allow us to pass to the limit when ε tends to 0. Thus, we consider the solution $w_\varepsilon(t,x)$ of (3.1). Let T, $T \geq 1$ be a fixed positive constant and $\tau_0 = \log(T)$. We define $W_\varepsilon(\tau, X)$, the vorticity obtained from w_ε by the change of variables (1.4) and (1.5). A short computation shows that W_ε satisfies the system

$$\begin{aligned}
&\partial_\tau \left(W_\varepsilon - \alpha_1 e^{-\tau}\Delta W_\varepsilon\right) + \varepsilon e^{-\tau}\Delta^2 W_\varepsilon - \mathcal{L}(W_\varepsilon) + U_\varepsilon.\nabla\left(W_\varepsilon - \alpha_1 e^{-\tau}\Delta W_\varepsilon\right) + \alpha_1 e^{-\tau}\Delta W_\varepsilon \\
&\qquad +\alpha_1 e^{-\tau}\frac{X}{2}.\nabla\Delta W_\varepsilon - \beta e^{-2\tau}\text{div }\left(|A_\varepsilon|^2\nabla W_\varepsilon\right) - \beta e^{-2\tau}\text{div }\left(\nabla\left(|A_\varepsilon|^2\right)\wedge A_\varepsilon\right) = 0, \\
&\text{div}\, U_\varepsilon = 0, \\
&W_{\varepsilon|\tau=\tau_0} = W_0,
\end{aligned} \tag{4.1}$$

where $\tau_0 = \log(T)$, U_ε is obtained from W_ε via the Biot-Savart law (2.10), $A_\varepsilon = \nabla U_\varepsilon + (\nabla U_\varepsilon)^t$ and we recall that

$$\mathcal{L}(W_\varepsilon) = \Delta W_\varepsilon + W_\varepsilon + \tfrac{X}{2}.\nabla W_\varepsilon.$$

By theorem 3.1, it is clear that there exists $\tau_\varepsilon > \tau_0$ such that

$$W_\varepsilon \in C^1\left((\tau_0, \tau_\varepsilon), H^1(2)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), H^3(2)\right).$$

We assume also that the initial datum $W_0 \in H^2(2)$ satisfies the assumption (1.8) of Theorem 1.1, for some $\gamma > 0$. Let $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(X)dX$, we write the following decompositions

$$\begin{aligned}
W_\varepsilon &= \eta G + f_\varepsilon, \\
U_\varepsilon &= \eta V + K_\varepsilon,
\end{aligned} \tag{4.2}$$

where G is the Oseen vortex sheet defined by (1.3) and V is the divergence free vector field obtained from G via the Biot-Savart law (2.10). Using the fact that $\mathcal{L}(G) = 0$, one has the equality

$$\begin{aligned}
&\partial_\tau\left(f_\varepsilon - \alpha_1 e^{-\tau}\Delta f_\varepsilon\right) + \varepsilon e^{-\tau}\Delta^2 f_\varepsilon - \mathcal{L}(f_\varepsilon) + K_\varepsilon.\nabla\left(f_\varepsilon - \alpha_1 e^{-\tau}\Delta f_\varepsilon\right) \\
&\quad +\eta V.\nabla\left(f_\varepsilon - \alpha_1 e^{-\tau}\Delta f_\varepsilon\right) + \eta K_\varepsilon.\nabla\left(G - \alpha_1 e^{-\tau}\Delta G\right) + \alpha_1 e^{-\tau}\Delta f_\varepsilon \\
&\qquad +\alpha_1 e^{-\tau}\tfrac{X}{2}.\nabla\Delta f_\varepsilon + \eta\alpha_1 e^{-\tau}\Delta G + \eta\alpha_1 e^{-\tau}\tfrac{X}{2}.\nabla\Delta G + \eta\varepsilon e^{-\tau}\Delta^2 G \\
&\qquad\quad -\beta e^{-2\tau}\text{div }\left(|A_\varepsilon|^2\nabla f_\varepsilon + \eta|A_\varepsilon|^2\nabla G\right) - \beta e^{-2\tau}\text{div }\left(\nabla\left(|A_\varepsilon|^2\right)\wedge A_\varepsilon\right) = 0.
\end{aligned} \tag{4.3}$$

Let $M = M(\alpha_1, \beta) > 2$ be a positive constant which will be made more precise later. Let $\tau_\varepsilon^* \in (\tau_0, \tau_\varepsilon]$ be the largest time (depending on M) such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$, the following inequality holds

$$\begin{aligned}\|W_\varepsilon(\tau)\|_{H^1}^2 + \alpha_1 e^{-\tau} \|\Delta W_\varepsilon(\tau)\|_{L^2}^2 + \left\||X|^2 W_\varepsilon(\tau)\right\|_{L^2}^2 \\ +\alpha_1^2 e^{-2\tau} \left\||X|^2 \Delta W_\varepsilon(\tau)\right\|_{L^2}^2 \leq M\gamma (1-\theta)^6 .\end{aligned} \tag{4.4}$$

To simplify the notations in the following computations, we assume that $0 < \gamma \leq 1$ and we take T sufficiently large so that $\dfrac{\alpha_1}{T} = \alpha_1 e^{-\tau_0} \leq 1$.

Since $W_\varepsilon \in C^0([\tau_0, \tau_\varepsilon), H^2(2))$ and the condition (1.8) holds, τ_ε^* is well defined. Furthermore, there exists a positive constant C independent of W_0 such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\eta^2 + \|f_\varepsilon\|_{H^1}^2 + \alpha_1 e^{-\tau} \|\Delta f_\varepsilon\|_{L^2}^2 + \left\||X|^2 f_\varepsilon\right\|_{L^2}^2 + \alpha_1^2 e^{-2\tau} \left\||X|^2 \Delta f_\varepsilon\right\|_{L^2}^2 \leq CM\gamma (1-\theta)^6 . \tag{4.5}$$

Indeed, using Cauchy-Schwartz inequality, we get

$$\begin{aligned}\eta &= \int_{\mathbb{R}^2} W_0(X) dX \\ &= \int_{\mathbb{R}^2} \frac{1+|X|^2}{1+|X|^2} W_0(X) dX \\ &\leq \left(\int_{\mathbb{R}^2} \frac{1}{\left(1+|X|^2\right)^2} dX\right)^{1/2} \left(\int_{\mathbb{R}^2} \left(1+|X|^2\right)^2 |W_0(X)|^2 dX\right)^{1/2} \\ &\leq C \|W_0\|_{L^2(2)} .\end{aligned}$$

Considering the decomposition (4.2) and the smoothness of G, we obtain the inequality (4.5).

To simplify the notations, in this section we write f instead of f_ε, W instead of W_ε, U instead of U_ε and K instead of K_ε.

The aim of this section is to show that the inequality (1.9) of Theorem 1.1 holds for the regularized solutions of the system (4.1), provided that the condition (1.8) is satisfied by W_0. To this end, we consider a fixed constant θ such that $0 < \theta < 1$ which is twice the rate of convergence of W to ηG in $H^2(2)$. In fact, we will show that, under the assumption (1.8), the decaying of f to 0 in $H^2(2)$ is equivalent to $e^{-\frac{\theta\tau}{2}}$. As it is explained in the introduction of this paper, the spectrum of $\mathcal{L}$ in $L^2(m)$ does not allow the rate of convergence to be better than $e^{-\frac{\tau}{2}}$.

In order to get the inequality (1.9), we construct in this section an energy functional $E = E(\tau)$ such that, for every $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$E(\tau) \sim \|f(\tau)\|^2_{H^2(2)},$$

and there exists a positive constant $C = C(\alpha_1, \beta, \theta, \gamma)$ such that, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\partial_\tau E(\tau) + \theta E(\tau) \leq Ce^{-\tau}. \tag{4.6}$$

This inequality will enable us to show that $\tau_\varepsilon^* = +\infty$ and obtain, by the application of Gronwall Lemma,

$$E(\tau) \leq Ce^{-\theta\tau}, \text{ for all } \tau \in [\tau_0, +\infty).$$

This functional is built as the sum of several intermediate energy functionals in various functions spaces, for which we perform convenient estimates.

4.1 Estimates in $\dot{H}^{-\frac{1+\theta}{2}}$

We start by performing an estimate of the solution of (4.3) in the homogeneous Sobolev space $\dot{H}^{-\frac{1+\theta}{2}}(\mathbb{R}^2)$. Combined with the other estimates, it will give us an estimate in the classical Sobolev space $H^{-\frac{1+\theta}{2}}(\mathbb{R}^2)$. The motivation to do this comes from the fact that the H^1-estimate that we will perform later (see Lemma 4.3) makes the term $\|u\|^2_{L^2}$ appear on the right hand side of our H^1-energy inequality. In order to absorb this term, we look for an estimate in a Sobolev space of negative order. To this end, due to Lemma 2.1 and the fact that $\displaystyle\int_{\mathbb{R}^2} f(X)dX = 0$, for $\frac{3}{4} \leq s < 1$, one can apply the operator $(-\Delta)^{-s}$ to the equality (4.3) and take the inner product of it with $(-\Delta)^{-s} f$. Through the computations that we will perform below, one can see that, in order to get the estimate (4.6), we have to choose at least $s = \frac{1+\theta}{2}$. Actually, since we have to absorb terms coming from the non-linear part of (4.3), it is more convenient to take $\frac{1+\theta}{2} < s < 1$, for instance $s = \frac{3+\theta}{4}$. In [47], the considered operator was $(-\Delta)^{-3/4}$, which implied the restriction $0 < \theta < \frac{1}{2}$.

The next lemma summarizes the computations needed when applying $(-\Delta)^{-s}$ to (4.3) and taking the L^2-scalar product of it with $(-\Delta)^{-s} f$.

Lemma 4.1 *Let $f \in H^3(2)$ such that $\int_{\mathbb{R}^2} f(X)dX = 0$, then, for all $\frac{1}{2} \leq s < 1$ the three following equalities hold.*

$$\begin{aligned}
&\left((-\Delta)^{-s}\left(\tfrac{X}{2}.\nabla f\right), (-\Delta)^{-s} f\right)_{L^2} = -\left(s+\tfrac{1}{2}\right)\left\|(-\Delta)^{-s} f\right\|_{L^2}^2, \\
&\left((-\Delta)^{-s}\left(\mathcal{L}(f)\right), (-\Delta)^{-s} f\right)_{L^2} = -\left\|(-\Delta)^{\frac{1}{2}-s} f\right\|_{L^2}^2 - \left(s-\tfrac{1}{2}\right)\left\|(-\Delta)^{-s} f\right\|_{L^2}^2, \\
&\left((-\Delta)^{-s}\left(\tfrac{X}{2}.\nabla\Delta f\right), (-\Delta)^{-s} f\right)_{L^2} = (s+1)\left\|(-\Delta)^{\frac{1}{2}-s} f\right\|_{L^2}^2.
\end{aligned} \tag{4.7}$$

Proof: Using Fourier variables, it is easy to see that

$$\widehat{\tfrac{X}{2}.\nabla f} = -\widehat{f} - \tfrac{\xi}{2}.\nabla\widehat{f}, \quad \text{and} \quad \widehat{\tfrac{X}{2}.\nabla\Delta f} = 2\left|\xi\right|^2\widehat{f} + \tfrac{\xi|\xi|^2}{2}\nabla\widehat{f}.$$

The proof of this lemma is then obtained through the Plancherel formula and direct computations.

□

In order to obtain a priori estimates of f in $\dot{H}^{-\frac{1+\theta}{2}}(\mathbb{R}^2)$, we define the functional

$$E_1(\tau) = \frac{1}{2}\left(\left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \alpha_1 e^{-\tau}\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2\right).$$

The estimate in $\dot{H}^{-\frac{1+\theta}{2}}$ of f under the condition (4.4) is given in the next lemma.

Lemma 4.2 *Let $W \in C^1\left((\tau_0, \tau_\varepsilon), H^1(2)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), H^3(2)\right)$ be the solution of (4.1) satisfying the inequality (4.4) for some $\gamma > 0$. There exist $\gamma_0 > 0$ and $T_0 \geq 1$ such that if $T \geq T_0$ and $\gamma \leq \gamma_0$, then, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$, E_1 satisfies the inequality*

$$\begin{aligned}
\partial_\tau E_1 + \theta E_1 + \left(1 + \frac{1-\theta}{4}\alpha_1 e^{-\tau}\right)\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2 &\leq CM^3\gamma\left(1-\theta\right)^2 e^{-2\tau} \\
&+ CM\gamma\left(1-\theta\right)^2\left(\|f\|_{L^2(2)}^2 + \|\nabla f\|^2 + \alpha_1^2 e^{-2\tau}\|\Delta f\|_{L^2(1)}^2\right),
\end{aligned} \tag{4.8}$$

where θ, $0 < \theta < 1$ is the fixed constant introduced at the beginning of Section 4.

Proof: Since $\int_{\mathbb{R}^2} f(X)dX = 0$, according to Lemma 2.1, $(-\Delta)^{-\frac{3+\theta}{4}} f$ is well defined. Thus, we apply $(-\Delta)^{-\frac{3+\theta}{4}}$ to the equality (4.3) and we get

$$\begin{aligned}
\partial_\tau\left((-\Delta)^{-\frac{3+\theta}{4}} f + \alpha_1 e^{-\tau}\left(-\Delta\right)^{\frac{1-\theta}{4}} f\right) + \varepsilon e^{-\tau}\left(-\Delta\right)^{\frac{5-\theta}{4}} f - (-\Delta)^{-\frac{3+\theta}{4}}\left(\mathcal{L}(f)\right) \\
-\alpha_1 e^{-\tau}\left(-\Delta\right)^{\frac{1-\theta}{4}} f + \alpha_1 e^{-\tau}\left(-\Delta\right)^{-\frac{3+\theta}{4}}\left(\tfrac{X}{2}.\nabla\Delta f\right) = H\left(\tau, G, f, W\right),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}H(\tau,G,f,W) = (-\Delta)^{-\frac{3+\theta}{4}}\Big(&-K.\nabla\left(f-\alpha_1 e^{-\tau}\Delta f\right)-\eta V.\nabla\left(f-\alpha_1 e^{-\tau}\Delta f\right)\\ &-\eta K.\nabla\left(G-\alpha_1 e^{-\tau}\Delta G\right)-\eta\alpha_1 e^{-\tau}\Delta G-\eta\alpha_1 e^{-\tau}\tfrac{X}{2}.\nabla\Delta G\\ &-\eta\varepsilon e^{-\tau}\Delta^2 G+\beta e^{-2\tau}\text{curl div }\left(|A|^2 A\right)\Big).\end{aligned}$$

Taking the L^2-scalar product of (4.9) with $(-\Delta)^{-\frac{3+\theta}{4}}$ and taking into account the equalities

$$\left(-(-\Delta)^{-\frac{3+\theta}{4}}\left(\mathcal{L}(f)\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}=\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2+\left(\tfrac{1+\theta}{4}\right)\left\|(-\Delta)^{-\frac{3+\theta}{4}}f\right\|^2,$$

and

$$\left(\alpha_1 e^{-\tau}(-\Delta)^{-\frac{3+\theta}{4}}\left(\tfrac{X}{2}.\nabla\Delta f\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}=\left(\tfrac{7+\theta}{4}\right)\alpha_1 e^{-\tau}\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2,$$

given by Lemma 4.1, we obtain

$$\begin{aligned}\tfrac{1}{2}\partial_\tau\left(\left\|(-\Delta)^{-\frac{3+\theta}{4}}f\right\|^2+\alpha_1 e^{-\tau}\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2\right)+\varepsilon e^{-\tau}\left\|(-\Delta)^{-\frac{1-\theta}{4}}f\right\|^2\\ +\left(\tfrac{1+\theta}{4}\right)\left\|(-\Delta)^{-\frac{3+\theta}{4}}f\right\|^2+\left(1+\left(\tfrac{1+\theta}{4}\right)\alpha_1 e^{-\tau}\right)\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2\\ =\left(H(\tau,G,f),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}.\end{aligned}\tag{4.10}$$

Now, it remains to estimate the right hand side of (4.10), that we write as

$$\left(H(\tau,G,f),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}=I_1+I_2+I_3+I_4+I_5,$$

where

$$\begin{aligned}I_1&=\left((-\Delta)^{-\frac{3+\theta}{4}}\left(-K.\nabla\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2},\\ I_2&=\left((-\Delta)^{-\frac{3+\theta}{4}}\left(-\eta V.\nabla\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2},\\ I_3&=\left((-\Delta)^{-\frac{3+\theta}{4}}\left(-\eta K.\nabla\left(G-\alpha_1 e^{-\tau}\Delta G\right)\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2},\\ I_4&=\left((-\Delta)^{-\frac{3+\theta}{4}}\left(-\eta\alpha_1 e^{-\tau}\Delta G-\eta\alpha_1 e^{-\tau}\tfrac{X}{2}.\nabla\Delta G-\eta\varepsilon e^{-\tau}\Delta^2 G\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}\\ &=\left((-\Delta)^{-\frac{3+\theta}{4}}J,(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2},\\ I_5&=\left((-\Delta)^{-\frac{3+\theta}{4}}\left(\beta e^{-2\tau}\text{curl div }\left(|A|^2 A\right)\right),(-\Delta)^{-\frac{3+\theta}{4}}f\right)_{L^2}.\end{aligned}$$

The remaining of the proof of this lemma is devoted to the estimate of these terms. We recall that $\operatorname{curl} K = f$, $\operatorname{curl} V = G$ and $\operatorname{curl} U = W$. Since the divergence of K vanishes, we obtain

$$\begin{aligned} I_1 &= \left((-\Delta)^{-\frac{3+\theta}{4}} \left(-\operatorname{div}\left(K\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)\right), (-\Delta)^{-\frac{3+\theta}{4}} f\right)_{L^2} \\ &\leq \left\|(-\Delta)^{-\frac{3+\theta}{4}} \nabla\left(K\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)\right\| \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|. \end{aligned}$$

Using the inequalities (2.1) of Lemma 2.1 and (2.15) of Lemma 2.4, together with the Young and Hölder inequalities and the property (4.5), we get

$$\begin{aligned} I_1 &\leq \frac{C}{(1-\theta)^{3/2}} \left\|K\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right\|_{L^2(1)} \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\| \\ &\leq \frac{C}{(1-\theta)^{3/2}} \|K\|_{L^\infty} \left\|f-\alpha_1 e^{-\tau}\Delta f\right\|_{L^2(1)} \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\| \\ &\leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{C}{\mu(1-\theta)^3} \|f\|_{L^2(2)} \|f\|_{H^1} \left\|f-\alpha_1 e^{-\tau}\Delta f\right\|^2_{L^2(1)} \\ &\leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{CM\gamma(1-\theta)^3}{\mu} \left(\|f\|^2_{L^2(1)} + \alpha_1^2 e^{-2\tau} \|\Delta f\|^2_{L^2(1)}\right), \end{aligned} \tag{4.11}$$

where μ is a positive constant which is made more precise later.

Similar computations and the inequality (4.5) give similar estimates for I_2. One has

$$I_2 \leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{CM\gamma(1-\theta)^3}{\mu} \left(\|f\|^2_{L^2(1)} + \alpha_1^2 e^{-2\tau} \|\Delta f\|^2_{L^2(1)}\right). \tag{4.12}$$

We likewise estimate the term I_3. Indeed, the same computations and the smoothness of G yield

$$\begin{aligned} I_3 &\leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{C\eta^2}{\mu(1-\theta)^3} \|f\|_{L^2(2)} \|f\|_{H^1} \left\|G-\alpha_1 e^{-\tau}\Delta G\right\|^2_{L^2(1)} \\ &\leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{CM\gamma(1-\theta)^3}{\mu} \left(\|f\|^2_{L^2(2)} + \|f\|^2_{H^1}\right)\left(1+\alpha_1 e^{-\tau}\right). \end{aligned}$$

Taking T_0 sufficiently large so that $\alpha_1 e^{-\tau} \leq 1$, we get

$$I_3 \leq \mu \left\|(-\Delta)^{-\frac{3+\theta}{4}} f\right\|^2 + \frac{CM\gamma(1-\theta)^3}{\mu} \left(\|f\|^2_{L^2(2)} + \|\nabla f\|^2\right). \tag{4.13}$$

Estimating I_4 is simple, because of the smoothness of G. We first remark the fact that $\int_{\mathbb{R}^2} J(X)dX = 0$. Thus we can apply the inequality (2.2) of Lemma 2.1 to obtain

$$\left\| (-\Delta)^{-\frac{3+\theta}{4}} J \right\| \leq \frac{C \left| \eta \right| e^{-\tau} \left\| G \right\|_{H^4(3)}}{(1-\theta)^{3/2}}.$$

Using the above inequality and the smoothness of G, we can write

$$\begin{aligned} I_4 &\leq \frac{C\eta e^{-\tau}}{(1-\theta)^{3/2}} \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\| \\ &\leq \mu \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\|^2 + \frac{CM\gamma \left(1-\theta\right)^3 e^{-2\tau}}{\mu}. \end{aligned} \tag{4.14}$$

It remains estimate the term I_5. The inequality (2.1) of Lemma 2.1 implies

$$\begin{aligned} I_5 &\leq \frac{C\beta e^{-2\tau}}{(1-\theta)^{3/2}} \left\| \nabla \left(\left| A \right|^2 A \right) \right\|_{L^2(1)} \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\| \\ &\leq \mu \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\|^2 + \frac{C\beta^2 e^{-4\tau}}{\mu \left(1-\theta\right)^3} \left\| \nabla \left(\left| A \right|^2 A \right) \right\|_{L^2(1)}^2. \end{aligned}$$

A short computation leads to

$$\left\| \nabla \left(\left| A \right|^2 A \right) \right\|_{L^2(1)}^2 \leq C \left\| \left| \nabla U \right|^2 \nabla^2 U \right\|_{L^2(1)}^2.$$

Using Hölder inequalities, the inequality (2.15) of Lemma 2.4 and the inequality (2.18) of Lemma 2.5, we get

$$\begin{aligned} \left\| \nabla \left(\left| A \right|^2 A \right) \right\|_{L^2(1)}^2 &\leq C \left\| \nabla U \right\|_{L^\infty}^4 \left\| \nabla^2 U \right\|_{L^2(1)}^2 \\ &\leq \left\| W \right\|_{L^2(2)}^2 \left\| W \right\|_{H^1}^2 \left(\left\| W \right\|_{L^2}^2 + \left\| \nabla W \right\|_{L^2(1)}^2 \right). \end{aligned}$$

Finally, taking into account the inequality (4.4), we get

$$\begin{aligned} I_5 &\leq \mu \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\|^2 + \frac{CM^3 \beta^2 \gamma^3 \left(1-\theta\right)^{18} e^{-4\tau}}{\mu} \left(1 + \frac{e^\tau}{\alpha_1} \right) \\ &\leq \mu \left\| (-\Delta)^{-\frac{3+\theta}{4}} f \right\|^2 + \frac{CM^3 \gamma^3 \left(1-\theta\right)^{18} e^{-3\tau}}{\mu}. \end{aligned} \tag{4.15}$$

The equality (4.10) and the inequalities (4.11), (4.12), (4.13), (4.14), (4.15) imply that

$$\begin{aligned}\frac{1}{2}\partial_\tau\left(\left\|(-\Delta)^{-\frac{3+\theta}{4}}f\right\|^2+\alpha_1e^{-\tau}\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2\right)+\left(\frac{1-20\mu+\theta}{4}\right)\left\|(-\Delta)^{-\frac{3+\theta}{4}}f\right\|^2\\+\left(1+\left(\frac{1+\theta}{4}\right)\alpha_1e^{-\tau}\right)\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2\\ \leq\frac{CM^3\gamma\left(1-\theta\right)^3e^{-2\tau}}{\mu}+\frac{CM\gamma\left(1-\theta\right)^3}{\mu}\left(\|f\|^2_{L^2(2)}+\|\nabla f\|^2+\alpha_1^2e^{-2\tau}\|\Delta f\|^2_{L^2(1)}\right).\end{aligned}\tag{4.16}$$

Setting $\mu=\dfrac{1-\theta}{20}$, we finally get

$$\begin{aligned}\partial_\tau E_1+\theta E_1+\left(1+\frac{1-\theta}{4}\alpha_1e^{-\tau}\right)\left\|(-\Delta)^{-\frac{1+\theta}{4}}f\right\|^2\leq CM^3\gamma\left(1-\theta\right)^2e^{-2\tau}\\+CM\gamma\left(1-\theta\right)^2\left(\|f\|^2_{L^2(2)}+\|\nabla f\|^2+\alpha_1^2e^{-2\tau}\|\Delta f\|^2_{L^2(1)}\right).\end{aligned}\tag{4.17}$$

□

4.2 Estimates in $H^1(\mathbb{R}^2)$

We next establish an H^1-estimate of f. As explained earlier, we get it by performing the L^2-scalar product of (4.3) with f. In this section, we will see how useful the lemma 4.2 is for absorbing bad terms which appear in the computations made below. One defines the functional

$$E_2(\tau)=\frac{1}{2}\left(\|f\|^2+\alpha_1e^{-\tau}\|\nabla f\|^2\right).$$

The H^1 estimate of f is given by the following lemma.

Lemma 4.3 *Let $W\in C^1\left((\tau_0,\tau_\varepsilon),H^1(2)\right)\cap C^0\left((\tau_0,\tau_\varepsilon),H^3(2)\right)$ be the solution of (4.1) satisfying the inequality (4.4) for some $\gamma>0$. There exist $\gamma_0>0$ and $T_0\geq 1$ such that if $T\geq T_0$ and $\gamma\leq\gamma_0$, then, for all $\tau\in[\tau_0,\tau_\varepsilon^*)$, E_2 satisfies the inequality*

$$\begin{aligned}\partial_\tau E_2+E_2+\tfrac{1}{2}\|\nabla f\|^2+\tfrac{\beta}{2}e^{-2\tau}\||A|\nabla f\|^2\\ \leq\|f\|^2+CM\gamma\left(1-\theta\right)^6\left(\|f\|^2+\left\||X|^2f\right\|^2\right)+CM^2\gamma\left(1-\theta\right)^6e^{-\tau},\end{aligned}\tag{4.18}$$

where θ, $0<\theta<1$ is the fixed constant introduced at the beginning of Section 4.

Proof: Taking the L^2-inner product of (4.3) with f, performing several integrations by parts and taking into account the equalities

$$(-\mathcal{L}(f), f)_{L^2} = \|\nabla f\|^2 - \frac{1}{2}\|f\|^2,$$

and

$$\alpha_1 e^{-\tau}\left(\frac{X}{2}.\nabla\Delta f, f\right)_{L^2} = \alpha_1 e^{-\tau}\|\nabla f\|^2,$$

we obtain the equality

$$\begin{aligned}\partial_\tau E_2 + E_2 + \varepsilon e^{-\tau}\|\Delta f\|^2 + (1-\alpha_1 e^{-\tau})\|\nabla f\|^2 + \beta e^{-2\tau}\||A|\nabla f\|^2 \\ = \|f\|^2 + I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
I_1 &= -\left(K.\nabla\left(f - \alpha_1 e^{-\tau}\Delta f\right), f\right)_{L^2}, \\
I_2 &= -\eta\left(K.\nabla\left(G - \alpha_1 e^{-\tau}\Delta G\right), f\right)_{L^2}, \\
I_3 &= -\eta\left(V.\nabla\left(f - \alpha_1 e^{-\tau}\Delta f\right), f\right)_{L^2}, \\
I_4 &= -\eta\alpha_1 e^{-\tau}\left(\frac{\varepsilon}{\alpha_1}\Delta^2 G + \Delta G + \frac{X}{2}.\nabla\Delta G, f\right)_{L^2} + \eta\beta e^{-2\tau}\left(\text{div }\left(|A|^2\nabla G\right), f\right)_{L^2}, \\
I_5 &= \left(\beta e^{-2\tau}\text{div }\left(\nabla\left(|A|^2\right)\wedge A\right), f\right)_{L^2}.
\end{aligned}$$

We notice that, since K is divergence free, $(K.\nabla f, f)_{L^2} = 0$. Integrating by parts and using the inequality (2.15) of lemma 2.4 and the inequality (4.5), we obtain

$$\begin{aligned}
I_1 &= -\alpha_1 e^{-\tau}(K\Delta f, \nabla f)_{L^2} \\
&\leq C\alpha_1 e^{-\tau}\|K\|_{L^\infty}\|\Delta f\|\|\nabla f\| \\
&\leq C\alpha_1 e^{-\tau}\|f\|_{H^1}^{1/2}\|f\|_{L^2(2)}^{1/2}\|\Delta f\|\|\nabla f\| \\
&\leq C\sqrt{\alpha_1}M\gamma(1-\theta)^6 e^{-\tau/2}\|\nabla f\| \\
&\leq \mu\|\nabla f\|^2 + \frac{CM^2\gamma^2(1-\theta)^{12}}{\mu}e^{-\tau},
\end{aligned} \tag{4.20}$$

where $\mu > 0$ will be made more precise later.

By the same method, using the inequality (2.14) of the lemma 2.4 and the smoothness of G, one has

$$
\begin{aligned}
I_2 &= \eta\left(K\left(G-\alpha_1 e^{-\tau}\Delta G\right),\nabla f\right)_{L^2}\\
&\leq |\eta|\,\|K\|_{L^4}\left\|G-\alpha_1 e^{-\tau}\Delta G\right\|_{L^4}\|\nabla f\|\\
&\leq C\left(1+\alpha_1 e^{-\tau}\right)|\eta|\,\|f\|_{L^2(2)}\|\nabla f\|\\
&\leq \mu\|\nabla f\|^2+\frac{CM\gamma(1-\theta)^6}{\mu}\left(\|f\|^2+\left\||X|^2 f\right\|^2\right).
\end{aligned}
\tag{4.21}
$$

The same method gives

$$
\begin{aligned}
I_3 &= -\alpha_1 e^{-\tau}\eta\left(V\Delta f,\nabla f\right)_{L^2}\\
&\leq \alpha_1 e^{-\tau}|\eta|\,\|V\|_{L^\infty}\|\Delta f\|\,\|\nabla f\|\\
&\leq C\sqrt{\alpha_1}M\gamma(1-\theta)^6 e^{-\tau/2}\|\nabla f\|\\
&\leq \mu\|\nabla f\|^2+\frac{CM^2\gamma^2(1-\theta)^{12}}{\mu}e^{-\tau}.
\end{aligned}
\tag{4.22}
$$

Because of the regularity of G, the estimate of I_4 is simple. Indeed, an integration by parts and Hölder inequalities yield

$$
\begin{aligned}
I_4 &\leq C|\eta|(\varepsilon+\alpha_1)e^{-\tau}\|f\|-\eta\beta e^{-2\tau}\left(|A|^2\nabla G,\nabla f\right)_{L^2}\\
&\leq C|\eta|(\varepsilon+\alpha_1)e^{-\tau}\|f\|+C|\eta|\beta e^{-2\tau}\|\nabla G\|_{L^\infty}\|\nabla U\|_{L^3}^2\|\nabla f\|_{L^3}
\end{aligned}
$$

Then, by the inequality (2.13), the continuous injection of $H^1(\mathbb{R}^2)$ into $L^3(\mathbb{R}^2)$ and the inequalities (4.4) and (4.5), one obtains

$$
\begin{aligned}
I_4 &\leq C|\eta|(\varepsilon+\alpha_1)e^{-\tau}\|f\|+C|\eta|\beta e^{-2\tau}\|W\|_{L^3}^2\|\nabla f\|_{L^3}\\
&\leq C(\varepsilon+\alpha_1)M\gamma(1-\theta)^6 e^{-\tau}+C|\eta|\beta e^{-2\tau}\|W\|_{H^1}^2\left(\|\nabla f\|+\|\Delta f\|\right)\\
&\leq C(\varepsilon+\alpha_1)M\gamma(1-\theta)^6 e^{-\tau}+CM^{3/2}\gamma^{3/2}(1-\theta)^9 e^{-\frac{3\tau}{2}}\\
&\leq CM^{3/2}\gamma(1-\theta)^6 e^{-\tau}.
\end{aligned}
\tag{4.23}
$$

Finally, using the same arguments, due to the inequality (2.13) and the continuous injection of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$, one has

$$
\begin{aligned}
I_5 &= -\beta e^{-2\tau}\left(\nabla\left(|A|^2\right)\wedge A,\nabla f\right)_{L^2}\\
&\leq C\beta e^{-2\tau}\|\nabla U\|_{L^4}\left\|\nabla^2 U\right\|_{L^4}\||A|\,\nabla f\|\\
&\leq C\beta e^{-2\tau}\|W\|_{H^1}\|W\|_{H^2}\||A|\,\nabla f\|\\
&\leq C\beta M\gamma(1-\theta)^6 e^{-2\tau}\frac{e^{\tau/2}}{\sqrt{\alpha_1}}\||A|\,\nabla f\|\\
&\leq \frac{\beta}{2}e^{-2\tau}\||A|\,\nabla f\|^2+CM^2\gamma^2(1-\theta)^{12}e^{-\tau}.
\end{aligned}
\tag{4.24}
$$

Taking into account the inequalities (4.20), (4.21), (4.22), (4.23) and (4.24) and assuming that $\gamma \leq 1$, we deduce from (4.19) that

$$\begin{aligned}\partial_\tau E_2 + E_2 + \left(1 - 3\mu - \alpha_1 e^{-\tau}\right) \|\nabla f\|^2 + \frac{\beta}{2} e^{-2\tau} \| |A| \nabla f\|^2 \leq \\ \|f\|^2 + \frac{CM\gamma\,(1-\theta)^6}{\mu} \left(\|f\|^2 + \left\||X|^2 f\right\|^2\right) + CM^2\gamma\,(1-\theta)^6 e^{-\tau},\end{aligned} \tag{4.25}$$

If we choose for instance $\mu = \frac{1}{12}$ and T_0 large enough to have $\alpha_1 e^{-\tau} \leq \frac{1}{4}$, we get

$$\begin{aligned}\partial_\tau E_2 + E_2 + \tfrac{1}{2} \|\nabla f\|^2 + \tfrac{\beta}{2} e^{-2\tau} \| |A| \nabla f\|^2 \leq \\ \|f\|^2 + CM\gamma\,(1-\theta)^6 \left(\|f\|^2 + \left\||X|^2 f\right\|^2\right) + CM^2\gamma\,(1-\theta)^6 e^{-\tau}.\end{aligned} \tag{4.26}$$

$\square$

To achieve the H^1-estimate of f, we have to combine the inequalities (4.8) and (4.18). We can interpolate $\|f\|^2$ between $\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2$ and $\|\nabla f\|^2$. Indeed, via Hölder and Young inequalities, we get

$$\begin{aligned}\|f\|^2 &= (2\pi)^2 \int_{\mathbb{R}^2} \frac{1}{|\xi|^{\frac{2(1+\theta)}{3+\theta}}} |\xi|^{\frac{2(1+\theta)}{3+\theta}} \left|\widehat{f}\right|^{\frac{2(1+\theta)}{3+\theta}} \left|\widehat{f}\right|^{\frac{4}{3+\theta}} d\xi \\ &\leq (2\pi)^2 \left(\int_{\mathbb{R}^2} \frac{1}{|\xi|^{1+\theta}} \left|\widehat{f}\right|^2 d\xi\right)^{\frac{2}{3+\theta}} \left(\int_{\mathbb{R}^2} |\xi|^2 \left|\widehat{f}\right|^2 d\xi\right)^{\frac{1+\theta}{3+\theta}} \\ &\leq \left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^{\frac{4}{3+\theta}} \|\nabla f\|^{\frac{2+2\theta}{3+\theta}} \\ &\leq \left(\frac{1+\theta}{3+\theta}\right) \frac{3}{8} \|\nabla f\|^2 + \left(\frac{2}{3+\theta}\right) \left(\frac{8}{3}\right)^{\frac{1+\theta}{2}} \left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2.\end{aligned}$$

Since, $0 < \theta < 1$, we obtain

$$\|f\|^2 \leq \frac{1}{4} \|\nabla f\|^2 + 5 \left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2. \tag{4.27}$$

Thus, we have

$$\begin{aligned}\partial_\tau E_2 + E_2 + \frac{1}{4} \|\nabla f\|^2 + \frac{\beta}{2} e^{-2\tau} \| |A| \nabla f\|^2 \leq \\ 5 \left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2 + CM\gamma\,(1-\theta)^6 \left(\|f\|^2 + \left\||X|^2 f\right\|^2\right) + CM^2\gamma\,(1-\theta)^6 e^{-\tau}.\end{aligned} \tag{4.28}$$

We define $E_3 = 6E_1 + E_2$. Inequalities (4.8) and (4.28) give

$$\partial_\tau E_3 + \theta E_3 + \left(1 + \frac{3}{2}(1-\theta)\alpha_1 e^{-\tau}\right)\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2 + \frac{1}{4}\|\nabla f\|^2 \leq CM^3\gamma(1-\theta)^2 e^{-\tau}$$
$$+CM\gamma(1-\theta)^2\left(\|f\|^2 + \|\nabla f\|^2 + \alpha_1^2 e^{-2\tau}\|\Delta f\|^2 + \left\||X|^2 f\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f\right\|^2\right). \tag{4.29}$$

Interpolating again $\|f\|^2$ between $\|\nabla f\|^2$ and $\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2$ and taking γ sufficiently small, we obtain

$$\partial_\tau E_3 + \theta E_3 + \frac{1}{2}\left\|(-\Delta)^{-\frac{1+\theta}{4}} f\right\|^2 + \frac{1}{8}\|\nabla f\|^2 \leq CM^3\gamma(1-\theta)^2 e^{-\tau}$$
$$+CM\gamma(1-\theta)^2\left(\alpha_1^2 e^{-2\tau}\|\Delta f\|^2 + \left\||X|^2 f\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f\right\|^2\right). \tag{4.30}$$

4.3 Estimates in $H^2(\mathbb{R}^2)$

We now perform a H^2 estimate of f. This is done with the same method as for the H^1 estimate in the previous section. Indeed, we perform the L^2 product between (4.3) and $-\Delta f$ and, after some computations, we see that the inequality (4.4) enables us to absorb all terms involving the H^2 norm of f. Combined with (4.30), we get an estimate in H^2, where only terms with weighted norms remain. More precisely, we introduce the following functional.

$$E_4(\tau) = \frac{1}{2}\left(\|\nabla f\|^2 + \alpha_1 e^{-\tau}\|\Delta f\|^2\right).$$

The H^2 estimate of f is given by the lemma below.

Lemma 4.4 *Let $W \in C^1((\tau_0,\tau_\varepsilon), H^1(2)) \cap C^0((\tau_0,\tau_\varepsilon), H^3(2))$ be the solution of* (4.1) *satisfying the inequality (4.4) for some $\gamma > 0$. There exist $\gamma_0 > 0$ and $T_0 \geq 1$ such that if $T \geq T_0$ and $\gamma \leq \gamma_0$, then for all $\tau \in [\tau_0, \tau_\varepsilon^*)$, E_4 satisfies the inequality*

$$\partial_\tau E_4 + E_4 + \frac{1}{2}\|\Delta f\|^2 + \frac{\beta}{2}e^{-2\tau}\left\||A|\Delta f\right\|^2 \leq \frac{3}{2}\|\nabla f\|^2 + CM^2\gamma(1-\theta)^6 e^{-2\tau}$$
$$+CM^2\gamma(1-\theta)^6\left(\|f\|^2 + \|\nabla f\|^2 + \left\||X|^2 f\right\|^2\right), \tag{4.31}$$

where θ, $0 < \theta < 1$ is the fixed constant introduced at the beginning of Section 4.

Proof: We take the L^2 product of (4.3) with $-\Delta f$. Doing several integrations by parts, it is easy to see that

$$(-\mathcal{L}(f), -\Delta f)_{L^2} = \|\Delta f\|^2 - \|\nabla f\|^2,$$

and

$$-\left(\alpha_1 e^{-\tau}\frac{X}{2}.\nabla\Delta f, \Delta f\right)_{L^2} = \frac{1}{2}\alpha_1 e^{-\tau}\|\Delta f\|^2.$$

Furthermore, one also has

$$\beta e^{-2\tau}\left(\text{div }\left(|A|^2\nabla f\right), \Delta f\right) = \beta e^{-2\tau}\||A|\,\Delta f\|^2 + \beta e^{-2\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2} A : \partial_j A \partial_j f \Delta f dX.$$

Using Hölder inequalities, the inequality (2.13) of lemma 2.3, the continuous injections of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$ and the inequality (4.4), we get

$$\begin{aligned}
\beta e^{-2\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2} A : \partial_j A\partial_j f\Delta f dX &\leq C\beta e^{-2\tau}\||A|\,\Delta f\|\,\|\nabla A\nabla f\| \\
&\leq C\beta e^{-2\tau}\||A|\,\Delta f\|\,\left\|\nabla^2 U\right\|_{L^4}\|\nabla f\|_{L^4} \\
&\leq C\beta e^{-2\tau}\||A|\,\Delta f\|\,\|\nabla W\|_{H^1}\|\nabla f\|_{H^1} \\
&\leq \mu_1\beta e^{-2\tau}\||A|\,\Delta f\|^2 \\
&\qquad + \frac{C\beta}{\mu_1}e^{-2\tau}\|W\|_{H^2}^2\left(\|\nabla f\|^2 + \|\Delta f\|^2\right) \\
&\leq \mu_1\beta e^{-2\tau}\||A|\,\Delta f\|^2 \\
&\qquad + \frac{CM\gamma\left(1-\theta\right)^6}{\mu_1}e^{-\tau}\left(\|\nabla f\|^2 + \|\Delta f\|^2\right),
\end{aligned}$$

where $\mu_1 > 0$ will be chosen later.

Consequently, we get

$$\begin{aligned}
\partial_\tau E_4 + \varepsilon e^{-\tau}\|\nabla\Delta f\|^2 + \left(1 - \frac{\alpha_1}{2}e^{-\tau}\right)\|\Delta f\|^2 + \beta\left(1-\mu_1\right)e^{-2\tau}\||A|\,\Delta f\|^2 \\
\leq \|\nabla f\|^2 + \frac{CM\gamma\left(1-\theta\right)^6}{\mu_1}e^{-\tau}\left(\|\nabla f\|^2 + \|\Delta f\|^2\right) + I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \tag{4.32}$$

where

$$
\begin{aligned}
I_1 &= \left(U.\nabla\left(f-\alpha_1 e^{-t}\Delta f\right), \Delta f\right)_{L^2}, \\
I_2 &= \eta\left(K.\nabla\left(G-\alpha_1 e^{-t}\Delta G\right), \Delta f\right)_{L^2}, \\
I_3 &= \eta\alpha_1 e^{-t}\left(\frac{\varepsilon}{\alpha_1}\Delta^2 G+\Delta G+\frac{X}{2}.\nabla\Delta G, \Delta f\right)_{L^2} + \eta\beta e^{-2t}\left(\text{div }\left(|A|^2\nabla G\right), \Delta f\right)_{L^2}, \\
I_4 &= \beta e^{-2t}\left(\text{div }\left(\nabla\left(|A|^2\right)\wedge A\right), \Delta f\right)_{L^2}.
\end{aligned}
$$

Integrating by parts and using the divergence free property of K, one can show that

$$
I_1 = -\sum_{j,k=1}^{2}\int_{\mathbb{R}^2}\partial_k U_j \partial_j f \partial_k f dX.
$$

Due to the Gagliardo-Niremberg inequality and the inequalities (2.13) and (4.4), it comes

$$
\begin{aligned}
I_1 \quad &\leq C\|\nabla U\|\|\nabla f\|_{L^4}^2 \\
&\leq C\|\nabla U\|\|\nabla f\|\|\Delta f\| \\
&\leq \mu_2\|\Delta f\|^2 + \frac{CM\gamma(1-\theta)^6}{\mu_2}\|\nabla f\|^2,
\end{aligned}
\tag{4.33}
$$

where $\mu_2 > 0$ will be chosen later.

We now estimate I_2 with the help of the inequality (2.15) of lemma 2.4, the inequality (4.5) and the smoothness of G.

$$
\begin{aligned}
I_2 \quad &\leq |\eta|\|K\|_{L^\infty}\left\|G-\alpha_1 e^{-\tau}\Delta G\right\|\|\Delta f\| \\
&\leq C|\eta|\left(1+\alpha_1 e^{-\tau}\right)\|f\|_{H^1}^{1/2}\|f\|_{L^2(2)}^{1/2}\|\Delta f\| \\
&\leq \mu_2\|\Delta f\|^2 + \frac{CM\gamma(1-\theta)^6}{\mu_2}\left(\|f\|^2+\|\nabla f\|^2+\left\||X|^2 f\right\|^2\right).
\end{aligned}
\tag{4.34}
$$

We rewrite

$$
I_3 = I_3^1 + I_3^2,
$$

where

$$
\begin{aligned}
I_3^1 &= \eta\alpha_1 e^{-\tau}\left(\frac{\varepsilon}{\alpha_1}\Delta^2 G+\Delta G+\frac{X}{2}.\nabla\Delta G, \Delta f\right)_{L^2}, \\
I_3^2 &= \eta\beta e^{-2\tau}\left(\text{div }\left(|A|^2\nabla G\right), \Delta f\right)_{L^2}.
\end{aligned}
$$

Using the good regularity of G and the inequality (4.5), one can show that

$$\begin{aligned} I_3^1 &\leq CM^{1/2}\gamma^{1/2}\left(1-\theta\right)^3 e^{-\tau}\left\|\Delta f\right\| \\ &\leq \mu_2\left\|\Delta f\right\|^2+\frac{CM\gamma\left(1-\theta\right)^6}{\mu_2}e^{-2\tau}. \end{aligned}$$

The estimate of I_3^2 is slightly more complicated. Actually, we can bound I_3^2 by two kinds of terms that we estimate separately. In fact, it is easy to see that

$$I_3^2 \leq C\left|\eta\right|\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|\nabla A\right|\left|A\right|\left|\nabla G\right|\left|\Delta f\right|dX+C\left|\eta\right|\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|A\right|^2\left|\nabla^2 G\right|\left|\Delta f\right|dX. \quad (4.35)$$

Each term of the right hand side of (4.35) can be estimated in a convenient way. We use again the inequality (2.13) of the lemma 2.3, inequality (4.4) , the Hölder inequalities and the inequality (4.4). We get

$$\begin{aligned} C\left|\eta\right|\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|\nabla A\right|\left|A\right|\left|\nabla G\right|\left|\Delta f\right|dX &\leq C\left|\eta\right|\beta e^{-2\tau}\left\|\nabla^2 U\right\|\left\|\nabla G\right\|_{L^\infty}\left\|\left|A\right|\Delta f\right\| \\ &\leq \mu_1\beta e^{-2\tau}\left\|\left|A\right|\Delta f\right\|^2+\frac{C\left|\eta\right|^2\beta}{\mu_1}e^{-2\tau}\left\|\nabla W\right\|^2 \\ &\leq \mu_1\beta e^{-2\tau}\left\|\left|A\right|\Delta f\right\|^2+\frac{CM^2\gamma^2\left(1-\theta\right)^{12}}{\mu_1}e^{-2\tau}. \end{aligned}$$

By the same way, we have

$$C\left|\eta\right|\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|A\right|^2\left|\nabla^2 G\right|\left|\Delta f\right|dx \leq \mu_1\beta e^{-2\tau}\left\|\left|A\right|\Delta f\right\|^2+\frac{CM^2\gamma^2\left(1-\theta\right)^{12}}{\mu_1}e^{-2\tau},$$

and thus we have shown

$$I_3^2 \leq 2\mu_1\beta e^{-2\tau}\left\|\left|A\right|\Delta f\right\|^2+\frac{CM^2\gamma^2\left(1-\theta\right)^{12}}{\mu_1}e^{-2\tau}.$$

Finally, assuming $\gamma \leq 1$, one has

$$I_3 \leq \mu_2\left\|\Delta f\right\|^2+2\mu_1\beta e^{-2\tau}\left\|\left|A\right|\Delta f\right\|^2+\frac{CM^2\gamma\left(1-\theta\right)^6}{\min(\mu_1,\mu_2)}e^{-2\tau}. \quad (4.36)$$

It remains to estimate I_4. Recalling that $U=\eta V+K$, one has

$$\begin{aligned} I_4 \quad &\leq C\left|\eta\right|\beta e^{-2\tau}\textstyle\int_{\mathbb{R}^2}\left|\nabla A\right|\left|A\right|\left|\nabla^2 V\right|\left|\Delta f\right|dX+C\left|\eta\right|\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|A\right|^2\left|\nabla^3 V\right|\left|\Delta f\right|dX \\ &+C\beta e^{-2\tau}\textstyle\int_{\mathbb{R}^2}\left|\nabla A\right|\left|A\right|\left|\nabla^2 K\right|\left|\Delta f\right|dX+C\beta e^{-2\tau}\int_{\mathbb{R}^2}\left|A\right|^2\left|\nabla^3 K\right|\left|\Delta f\right|dX. \end{aligned} \quad (4.37)$$

We have to estimate each term of the right hand side of the equality (4.37). The first two ones can be estimated exactly like we did for I_3^2. The inequality (2.13) of lemma 2.3 and Gagliardo-Niremberg inequality yield

$$\begin{aligned}
C\beta e^{-2\tau}\int_{\mathbb{R}^2}|\nabla A|\,|A|\left|\nabla^2 K\right||\Delta f|\,dX &\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2 \\
&\quad+\frac{C\beta e^{-2\tau}}{\mu_1}\left\|\nabla^2 U\right\|_{L^4}^2\left\|\nabla^2 K\right\|_{L^4}^2 \\
&\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2 \\
&\quad+\frac{C\beta e^{-2\tau}}{\mu_1}\|\nabla W\|_{L^4}^2\,\|\nabla f\|_{L^4}^2 \\
&\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2 \\
&\quad+\frac{C\beta e^{-2\tau}}{\mu_1}\|\nabla W\|\,\|\Delta W\|\,\|\nabla f\|\,\|\Delta f\|\,.
\end{aligned}$$

Due to the inequality (4.4), we get

$$\begin{aligned}
C\beta e^{-2\tau}\int_{\mathbb{R}^2}|\nabla A|\,|A|\left|\nabla^2 K\right||\Delta f|\,dX &\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2 \\
&\quad+\frac{CM\gamma\,(1-\theta)^6\,e^{-\frac{3\tau}{2}}}{\mu_1}\left(\|\nabla f\|^2+\|\Delta f\|^2\right).
\end{aligned}$$

By the same method, we obtain

$$\begin{aligned}
C\beta e^{-2\tau}\int_{\mathbb{R}^2}|A|^2\left|\nabla^3 K\right||\Delta f|\,dX &\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2+\frac{C\beta e^{-2\tau}}{\mu_1}\|\nabla U\|_{L^\infty}^2\left\|\nabla^3 K\right\|^2 \\
&\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2 \\
&\quad+\frac{C\beta e^{-2\tau}}{\mu_1}\|\nabla W\|_{H^1}\,\|\nabla W\|_{L^2(2)}\,\|\Delta f\|^2 \\
&\leq \mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2+\frac{CM\gamma\,(1-\theta)^6\,e^{-\tau}}{\mu_1}\|\Delta f\|^2\,.
\end{aligned}$$

Finally, we have shown that

$$I_4 \leq 4\mu_1\beta e^{-2\tau}\,\||A|\,\Delta f\|^2+\frac{CM^2\gamma\,(1-\theta)^6\,e^{-\tau}}{\mu_1}\left(\|\nabla f\|^2+\|\Delta f\|^2\right). \tag{4.38}$$

Going back to (4.32) and taking into account the inequalities (4.33), (4.34), (4.36) and (4.38), we get

$$\partial_\tau E_4 + \left(1 - 3\mu_2 - \frac{\alpha_1}{2} e^{-\tau}\right) \|\Delta f\|^2 + (1 - 7\mu_1)\,\beta e^{-2\tau} \left\| |A|\,\Delta f \right\|^2 \le \|\nabla f\|^2 + \frac{CM^2\gamma\,(1-\theta)^6}{\min(\mu_1,\mu_2)} \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 + \left\| |X|^2 f \right\|^2 \right) + \frac{CM^2\gamma\,(1-\theta)^6}{\min(\mu_1,\mu_2)} e^{-2\tau}.$$

Taking for instance $\mu_1 = \frac{1}{14}$, $\mu_2 = \frac{1}{12}$, γ small enough and $T = e^{\tau_0}$ large enough, we finally have

$$\partial_\tau E_4 + E_4 + \frac{1}{2} \|\Delta f\|^2 + \frac{\beta}{2} e^{-2\tau} \left\| |A|\,\Delta f \right\|^2 \le \frac{3}{2} \|\nabla f\|^2 + CM^2\gamma\,(1-\theta)^6 e^{-2\tau} + CM^2\gamma\,(1-\theta)^6 \left(\|f\|^2 + \|\nabla f\|^2 + \left\| |X|^2 f \right\|^2 \right). \tag{4.39}$$

□

In order to finish the H^2 estimate of f we define a new functional E_5 as a linear combination of E_3 and E_4 given by

$$E_5 = 16E_3 + E_4.$$

From the inequalities (4.30) and (4.31), it is clear that one has

$$\partial_\tau E_5 + \theta E_5 + 8 \left\| (-\Delta)^{-\frac{1+\theta}{4}} f \right\|^2 + \frac{1}{2} \|\nabla f\|^2 + \frac{1}{2} \|\Delta f\|^2 \le CM^3\gamma\,(1-\theta)^2 e^{-\tau} + CM^2\gamma\,(1-\theta)^2 \Big(\|f\|^2 + \|\nabla f\|^2 + \alpha_1^2 e^{-2\tau} \|\Delta f\|^2 + \left\| |X|^2 f \right\|^2 + \alpha_1^2 e^{-2\tau} \left\| |X|^2 \Delta f \right\|^2 \Big). \tag{4.40}$$

Using the interpolation inequality (4.27) and taking γ small enough and $\tau_0 = \log(T)$ large enough, we finally obtain

$$\partial_\tau E_5 + \theta E_5 + 7 \left\| (-\Delta)^{-\frac{1+\theta}{4}} f \right\|^2 + \frac{1}{4} \|\nabla f\|^2 + \frac{1}{4} \|\Delta f\|^2 \le CM^3\gamma\,(1-\theta)^2 e^{-\tau} + CM^2\gamma\,(1-\theta)^2 \left(\left\| |X|^2 f \right\|^2 + \alpha_1^2 e^{-2\tau} \left\| |X|^2 \Delta f \right\|^2 \right). \tag{4.41}$$

4.4 Estimates in $H^2(2)$

In order to achieve the estimate of f in $H^2(2)$, it remains to perform estimates in weighted spaces. Combined with the inequality (4.41), it will give us an estimate in $H^2(2)$. To do this, we make the L^2-scalar product of (4.3) with $|X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f\right)$. We define the functional

$$E_6(\tau) = \frac{1}{2} \left\| |X|^2 \left(f - \alpha_1 e^{-\tau} \Delta f\right) \right\|^2 .$$

Before stating the lemma which contains the estimate of E, we state a technical lemma, which gives the terms provided by the L^2-product of the linear terms of (4.3) with $|X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f\right)$.

Lemma 4.5 *Let $f \in C^1\left((\tau_0, \tau_\varepsilon), H^1(2)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), H^3(2)\right)$ and H be defined by $H(X,\tau,f) = |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f\right)$. For all $\tau \in (\tau_0, \tau_\varepsilon)$, the next equalities hold.*

1. $(-f, H(X,\tau,f))_{L^2} = -\left\| |X|^2 f \right\|^2 + 8\alpha_1 e^{-\tau} \left\| |X| f \right\|^2 - \alpha_1 e^{-\tau} \left\| |X|^2 \nabla f \right\|^2.$
2. $(-\Delta f, H(X,\tau,f))_{L^2} = \alpha_1 e^{-\tau} \left\| |X|^2 \Delta f \right\|^2 - 8 \left\| |X| f \right\|^2 + \left\| |X|^2 \nabla f \right\|^2 .$
3. $$\begin{aligned}\left(-\tfrac{X}{2}.\nabla f, H(X,\tau,f)\right)_{L^2} &= \tfrac{3}{2} \left\| |X|^2 f \right\|^2 - 24\alpha_1 e^{-\tau} \left\| |X| f \right\|^2 \\ &\quad + 3\alpha_1 e^{-\tau} \left\| |X|^2 \nabla f \right\|^2 - \tfrac{\alpha_1}{2} e^{-\tau} \left(X.\nabla \Delta f, |X|^4 f\right)_{L^2} .\end{aligned}$$
4. $$\begin{aligned}(-\mathcal{L}(f), H(X,\tau,f))_{L^2} &= \tfrac{1}{2} \left\| |X|^2 f \right\|^2 + (1 + 2\alpha_1 e^{-\tau}) \left\| |X|^2 \nabla f \right\|^2 \\ &\quad + \alpha_1 e^{-\tau} \left\| |X|^2 \Delta f \right\|^2 - (8 + 16\alpha_1 e^{-\tau}) \left\| |X| f \right\|^2 - \tfrac{\alpha_1}{2} e^{-\tau} \left(X.\nabla \Delta f, |X|^4 f\right)_{L^2} .\end{aligned}$$
5. $\left(\alpha_1 e^{-\tau} \frac{X}{2}.\nabla \Delta f, H(X,\tau,f)\right)_{L^2} = \frac{\alpha_1}{2} e^{-\tau} \left(X.\nabla \Delta f, |X|^4 f\right)_{L^2} + \frac{3\alpha_1^2}{2} e^{-2\tau} \left\| |X|^2 \Delta f \right\|^2 .$
6. $$\begin{aligned}\varepsilon e^{-\tau} \left(\Delta^2 f, H(X,\tau,f)\right)_{L^2} &= \varepsilon \alpha_1 e^{-2\tau} \left(\left\| |X|^2 \nabla \Delta f \right\|^2 - 8 \left\| |X| \Delta f \right\|^2 \right) \\ &\quad + \varepsilon e^{-\tau} \left(\left\| |X|^2 \Delta f \right\|^2 - 8 \left\| |X| \nabla f \right\|^2 + 32 \left\| f \right\|^2 - 16 \left\| X.\nabla f \right\|^2 \right) .\end{aligned}$$

Proof: All these equalities are obtained via integrations by parts. We only show the first four ones, the others are obtained with the same method. Let us show the equality 1. Two integrations by parts imply

$$\begin{aligned}\left(-f, |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f\right)\right)_{L^2} &= -\left\| |X|^2 f \right\|^2 - \alpha_1 e^{-\tau} \left\| |X|^2 \nabla f \right\|^2 \\ &\qquad - 4\alpha_1 e^{-\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} X_j |X|^2 f \partial_j f dX \\ &= -\left\| |X|^2 f \right\|^2 - \alpha_1 e^{-\tau} \left\| |X|^2 \nabla f \right\|^2 \\ &\qquad - 2\alpha_1 e^{-\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} X_j |X|^2 \partial_j \left(f^2\right) dX \\ &= -\left\| |X|^2 f \right\|^2 - \alpha_1 e^{-\tau} \left\| |X|^2 \nabla f \right\|^2 + 8\alpha_1 e^{-\tau} \left\| |X| f \right\|^2 .\end{aligned}$$

The equality 2. is obtained through the same computations. We show now the third equality of this lemma. Integrating by parts, we obtain

$$\begin{aligned}\left(-\frac{X}{2}.\nabla f, |X|^4 \left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)_{L^2} &= -\sum_{j=1}^{2}\int_{\mathbb{R}^2}\frac{X_j |X|^4}{4}\partial_j\left(|f|^2\right)dX \\ &\quad + \alpha_1 e^{-\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2}\frac{X_j |X|^4}{2}\partial_j f\Delta f dX \\ &= \frac{3}{2}\left\||X|^2 f\right\|^2 + \alpha_1 e^{-\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2}\frac{X_j |X|^4}{2}\partial_j f\Delta f dX.\end{aligned}$$

Besides, integrating several times by parts, we get

$$\begin{aligned}\alpha_1 e^{-\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2}\frac{X_j |X|^4}{2}\partial_j f\Delta f dX &= -\alpha_1 e^{-\tau}\sum_{j=1}^{2}\int_{\mathbb{R}^2} f\partial_j\left(\frac{X_j |X|^4}{2}\Delta f\right)dX \\ &= -3\alpha_1 e^{-\tau}\int_{\mathbb{R}^2}|X|^4 f\Delta f dX \\ &\quad - \frac{\alpha_1}{2}e^{-\tau}\left(X.\nabla\Delta f, |X|^4 f\right)_{L^2} \\ &= -24\alpha_1 e^{-\tau}\left\||X| f\right\|^2 + 3\alpha_1 e^{-\tau}\left\||X|^2\nabla f\right\|^2 \\ &\quad - \frac{\alpha_1}{2}e^{-\tau}\left(X.\nabla\Delta f, |X|^4 f\right)_{L^2},\end{aligned}$$

and consequently

$$\begin{aligned}\left(-\tfrac{X}{2}.\nabla f, |X|^4 \left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)_{L^2} &= \tfrac{3}{2}\left\||X|^2 f\right\|^2 - 24\alpha_1 e^{-\tau}\left\||X| f\right\|^2 \\ &\quad + 3\alpha_1 e^{-\tau}\left\||X|^2\nabla f\right\|^2 - \tfrac{\alpha_1}{2}e^{-\tau}\left(X.\nabla\Delta f, |X|^4 f\right)_{L^2}.\end{aligned}$$

The fourth equality of this lemma is obtained by summing the first three ones. By the same method, we obtain easily the equalities 5. and 6. of this lemma.

□

The $H^2(2)$ estimate of f is given in the following lemma.

Lemma 4.6 *Let* $W \in C^1((\tau_0, \tau_\varepsilon), H^1(2)) \cap C^0((\tau_0, \tau_\varepsilon), H^3(2))$ *be the solution of* (4.1) *satisfying the inequality (4.4) for some* $\gamma > 0$. *There exist* $\gamma_0 > 0$ *and* $T_0 \geq 1$ *such that if* $T \geq T_0$ *and* $\gamma \leq \gamma_0$, *then for all* $\tau \in [\tau_0, \tau_\varepsilon^*)$, E_6 *satisfies the inequality*

$$\partial_\tau E_6 + \theta E_6 + \frac{1-\theta}{8} \left\| |X|^2 f \right\|^2 + \frac{1}{4} \left\| |X|^2 \nabla f \right\|^2 + \frac{\alpha_1}{4} e^{-\tau} \left\| |X|^2 \Delta f \right\|^2$$
$$\leq CM^2 \gamma (1-\theta)^6 e^{-\tau} + \frac{1024}{1-\theta} \|f\|^2 + CM^2 \gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 \right), \tag{4.42}$$

where θ, $0 < \theta < 1$ *is the fixed constant introduced at the beginning of Section 4.*

Proof: To show this lemma, we perform the L^2-product of the equality (4.3) with $|X|^4 (f - \alpha_1 e^{-\tau} \Delta f)$. Applying the lemma 4.5, we obtain

$$\partial_\tau E_6 + \frac{1}{2} \left\| |X|^2 f \right\|^2 + \left(1 + \alpha_1 e^{-\tau}\right) \left\| |X|^2 \nabla f \right\|^2 + \left(\alpha_1 e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau} \right) \left\| |X|^2 \Delta f \right\|^2 + J$$
$$= C\varepsilon e^{-\tau} \| |X| \nabla f\|^2 + C\varepsilon \alpha_1 e^{-2\tau} \| |X| \Delta f\|^2 + \left(8 + 8\alpha_1 e^{-\tau}\right) \| |X| f\|^2$$
$$+ I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.43}$$

where

$$J = -\beta e^{-2\tau} \left(\mathrm{div}\, \left(|A|^2 \nabla f \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2},$$
$$I_1 = \left(K.\nabla \left(f - \alpha_1 e^{-\tau} \Delta f \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2},$$
$$I_2 = \eta \left(K.\nabla \left(G - \alpha_1 e^{-\tau} \Delta G \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2},$$
$$I_3 = \eta \left(V.\nabla \left(f - \alpha_1 e^{-\tau} \Delta f \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2},$$
$$I_4 = -\eta \varepsilon e^{-\tau} \left(\Delta^2 G, |X|^2 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2}$$
$$+ \eta \alpha_1 e^{-\tau} \left(\Delta G + \frac{X}{2}.\nabla \Delta G, |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2}$$
$$- \eta \beta e^{-2\tau} \left(\mathrm{div}\, \left(|A|^2 \nabla G \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2},$$
$$I_5 = -\beta e^{-2\tau} \left(\mathrm{div}\, \left(\nabla \left(|A|^2 \right) \wedge A \right), |X|^4 \left(f - \alpha_1 e^{-\tau} \Delta f \right) \right)_{L^2}.$$

We estimate now J. One has

$$J = J_1 + J_2, \tag{4.44}$$

where

$$J_1 = -\beta e^{-2\tau} \left(\text{div } \left(|A|^2 \nabla f\right), |X|^4 f\right)_{L^2},$$
$$J_2 = \beta\alpha_1 e^{-3\tau} \left(\text{div } \left(|A|^2 \nabla f\right), |X|^4 \Delta f\right)\right)_{L^2}.$$

We estimate J_1 and J_2 separately. Integrating by parts, we obtain

$$J_1 = \beta e^{-2\tau} \left\||X|^2 |A| \nabla f\right\|^2 + 4\beta e^{-2\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} X_j |X|^2 |A|^2 \partial_j f f dX.$$

Using Hölder and Young inequalities, we obtain

$$\left|4\beta e^{-2\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} X_j |X|^2 |A|^2 \partial_j f f dX\right| \leq \frac{\beta}{2} e^{-2\tau} \left\||X|^2 |A| \nabla f\right\|^2 + C\beta e^{-2\tau} \left\||X| f\right\|^2 \|\nabla U\|_{L^\infty}^2.$$

Then, using the inequality (2.15) of Lemma 2.4, the inequality (2.4) of Lemma 2.2 and the conditions (4.4) and (4.5), we get

$$\begin{aligned}\left|4\beta e^{-2\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} X_j |X|^2 |A|^2 \partial_j f f dX\right| &\leq \frac{\beta}{2} e^{-2\tau} \left\||X|^2 |A| \nabla f\right\|^2 \\ &\quad + C\beta e^{-2\tau} \left\||X|^2 f\right\| \|f\| \|\nabla W\|_{H^1} \|\nabla W\|_{L^2(2)} \\ &\leq \frac{\beta}{2} e^{-2\tau} \left\||X|^2 |A| \nabla f\right\|^2 + CM^2\gamma^2 (1-\theta)^{12} e^{-\tau},\end{aligned}$$

and we conclude that

$$J_1 \geq \frac{\beta}{2} e^{-2\tau} \left\||X|^2 |A| \nabla f\right\|^2 - CM^2\gamma^2 (1-\theta)^{12} e^{-\tau}. \tag{4.45}$$

By the same way, we estimate J_2. A short computation shows that

$$J_2 = \beta\alpha_1 e^{-3\tau} \left\||X|^2 |A| \Delta f\right\|^2 + 2\beta\alpha_1 e^{-3\tau} \textstyle\sum_{j=1}^{2} \int_{\mathbb{R}^2} |X|^4 \partial_j A : A \partial_j f \Delta f dX.$$

We define

$$I = \left|2\beta\alpha_1 e^{-3\tau} \sum_{j=1}^{2} \int_{\mathbb{R}^2} |X|^4 \partial_j A : A \partial_j f \Delta f dX\right|$$

Applying Hölder inequalities and the continuous injection of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$, we obtain

$$\begin{aligned} I &\leq C\beta\alpha_1 e^{-3\tau} \left\| |X|^2 |A| \Delta f \right\| \left\| |X|^2 \nabla f \right\|_{L^4} \left\| \nabla^2 U \right\|_{L^4} \\ &\leq C\beta\alpha_1 e^{-3\tau} \left\| |X|^2 |A| \Delta f \right\| \left\| |X|^2 \nabla f \right\|_{L^4} \left\| \nabla^2 U \right\|_{H^1} . \end{aligned}$$

Using the inequality (2.6) of Lemma 2.2 and the inequality (2.17) of Lemma 2.4, we get

$$\begin{aligned} I \leq C\beta\alpha_1 e^{-3\tau} & \left\| |X|^2 |A| \Delta f \right\| \left\| |X|^2 \nabla f \right\|^{1/2} \\ & \times \left(\|f\|^{1/2} + \left\| |X| \nabla f \right\|^{1/2} + \left\| |X|^2 \Delta f \right\|^{1/2} \right) \|W\|_{H^2} . \end{aligned}$$

Due to Young inequality and the condition (4.4), we obtain

$$\begin{aligned} I &\leq \frac{\beta}{2}\alpha_1 e^{-3\tau} \left\| |X|^2 |A| \Delta f \right\|^2 \\ &\quad + C\beta\alpha_1 e^{-3\tau} \|W\|_{H^2}^2 \left(\|f\|^2 + \|\nabla f\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) \\ &\leq \frac{\beta}{2}\alpha_1 e^{-3\tau} \left\| |X|^2 |A| \Delta f \right\|^2 \\ &\quad + CM\gamma (1-\theta)^6 e^{-2\tau} \left(\|f\|^2 + \|\nabla f\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) . \end{aligned}$$

Thus, we can conclude that

$$\begin{aligned} J_2 \geq \frac{\beta}{2}\alpha_1 e^{-3\tau} & \left\| |X|^2 |A| \Delta f \right\|^2 \\ & - CM\gamma (1-\theta)^6 e^{-2\tau} \left(\|f\|^2 + \|\nabla f\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) . \end{aligned} \tag{4.46}$$

Combining the inequalities (4.45) and (4.46) and going back to (4.44), we have shown that

$$\begin{aligned} J \geq \frac{\beta}{2} e^{-2\tau} & \left(\left\| |X|^2 |A| \nabla f \right\|^2 + \alpha_1 e^{-\tau} \left\| |X|^2 |A| \Delta f \right\|^2 \right) - CM^2\gamma^2 (1-\theta)^{12} e^{-\tau} \\ & - CM\gamma (1-\theta)^6 e^{-2\tau} \left(\|f\|^2 + \|\nabla f\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) . \end{aligned} \tag{4.47}$$

Taking into account the inequality (4.47), the equality (4.43) becomes

$$\begin{aligned}\partial_\tau E_6 + \frac{1}{2}\left\||X|^2 f\right\|^2 + \left(1+\alpha_1 e^{-\tau}\right)\left\||X|^2 \nabla f\right\|^2 + \left(\alpha_1 e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau}\right)\left\||X|^2 \Delta f\right\|^2 \\ + \frac{\beta}{2} e^{-2\tau}\left(\left\||X|^2 |A| \nabla f\right\|^2 + \alpha_1 e^{-\tau}\left\||X|^2 |A| \Delta f\right\|^2\right) \leq \\ C\varepsilon e^{-\tau}\left\||X| \nabla f\right\|^2 + C\varepsilon\alpha_1 e^{-2\tau}\left\||X| \Delta f\right\|^2 + \left(8+8\alpha_1 e^{-\tau}\right)\left\||X| f\right\|^2 \\ + CM\gamma(1-\theta)^6 e^{-2\tau}\left(\|f\|^2 + \|\nabla f\|^2 + \left\||X|^2 \nabla f\right\|^2 + \left\||X|^2 \Delta f\right\|^2\right) \\ + CM^2\gamma^2 (1-\theta)^{12} e^{-\tau} + I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{4.48}$$

It remains to estimate every I_i, $i = 1, ..., 5$. Using the divergence free property of K, integrating by parts and using Hölder inequalities, we get

$$\begin{aligned} I_1 &= -2\sum_{j=1}^{2}\int_{\mathbb{R}^2} X_j |X|^2 K_j \left|f-\alpha_1 e^{-\tau}\Delta f\right|^2 dX \\ &\leq C\|K\|_{L^\infty}\left\||X|^2\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right\|\left\||X|\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right\|. \end{aligned}$$

The inequalities (2.15) of lemma 2.3 and (2.4) of lemma 2.2, the Young inequality $ab \leq \frac{3}{4}a^{\frac{4}{3}} + \frac{1}{4}b^4$ and the inequality (4.5) yield

$$\begin{aligned} I_1 &\leq C\|f\|_{H^1}^{1/2}\|f\|_{L^2(2)}^{1/2}\left\||X|^2\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right\|^{3/2}\left\|f-\alpha_1 e^{-\tau}\Delta f\right\|^{1/2} \\ &\leq CM^{1/2}\gamma^{1/2}(1-\theta)^3\left(\left\||X|^2\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right\|^2 + \left\|f-\alpha_1 e^{-\tau}\Delta f\right\|^2\right) \\ &\leq CM^{1/2}\gamma^{1/2}(1-\theta)^3\left(\left\||X|^2 f\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f\right\|^2 + \|f\|^2 + \alpha_1^2 e^{-2\tau}\|\Delta f\|^2\right). \end{aligned} \tag{4.49}$$

Using the inequality (2.14) of Lemma 2.4, one can bound I_2 in a convenient way. Indeed, one has

$$\begin{aligned} I_2 &\leq C|\eta|\,\|K\|_{L^4}\left\||X|^2\nabla\left(G-\alpha_1 e^{-\tau}\Delta G\right)\right\|_{L^4}\left\||X|^2\left(f-\alpha_1 e^{-t}\Delta f\right)\right\| \\ &\leq C|\eta|\,\|f\|_{L^2(2)}\left(\left\||X|^2 f\right\| + \alpha_1 e^{-\tau}\left\||X|^2 \Delta f\right\|\right) \\ &\leq CM^{1/2}\gamma^{1/2}(1-\theta)^3\left(\left\||X|^2 f\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f\right\|^2 + \|f\|^2\right). \end{aligned} \tag{4.50}$$

Via an integration by parts, due to the facts that $V(X).X = 0$ and $\operatorname{div} V = 0$, we show that I_3 vanishes. Indeed

$$\begin{aligned} I_3 &= \frac{\eta}{2}\sum_{j=1}^{2}\int_{\mathbb{R}^2}|X|^4 V_j\partial_j\left(\left|f-\alpha_1 e^{-\tau}\Delta f\right|^2\right)dX \\ &= -2\eta\sum_{j=1}^{2}\int_{\mathbb{R}^2}|X|\,X_jV_j\left|f-\alpha_1 e^{-\tau}\Delta f\right|^2 dX = 0. \end{aligned}$$

We rewrite $I_4 = I_4^1 + I_4^2$, where

$$\begin{aligned} I_4^1 &= -\eta\alpha_1 e^{-\tau}\left(\frac{\varepsilon}{\alpha_1}\Delta^2 G + \Delta G + \frac{X}{2}.\nabla\Delta G, |X|^2\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)_{L^2}, \\ I_4^2 &= -\eta\beta e^{-2\tau}\left(\operatorname{div}\left(|A|^2\nabla G\right), |X|^4\left(f-\alpha_1 e^{-\tau}\Delta f\right)\right)_{L^2}. \end{aligned}$$

It is easy, using the smoothness of G and the inequality (4.5), to see that

$$\begin{aligned} I_4^1 &\leq C\,|\eta|\,e^{-\tau}\,\|G\|_{H^3(3)}\left(\|f\| + \alpha_1 e^{-\tau}\|\Delta f\|\right) \\ &\leq CM\gamma\,(1-\theta)^6\,e^{-\tau}. \end{aligned}$$

The term I_4^3 is not really harder to estimate. Due to the inequality (2.13) of Lemma 2.3, the inequality (4.5), the continuous injection of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$ and the inequality (4.4), we get

$$\begin{aligned} I_4^2 &\leq |\eta|\,\beta e^{-2\tau}\left(\|\nabla U\|_{L^4}^2\left\||X|^4\Delta G\right\|_{L^\infty} + \|\nabla U\|_{L^4}\left\|\nabla^2 U\right\|_{L^4}\left\||X|^4\nabla G\right\|_{L^\infty}\right) \\ &\qquad\qquad\qquad\qquad\times\left(\|f\| + \alpha_1 e^{-\tau}\|\Delta f\|\right) \\ &\leq C\,|\eta|\,\beta e^{-2\tau}\left(\|W\|_{L^4}^2 + \|W\|_{L^4}\|\nabla W\|_{L^4}\right)\left(\|f\| + \alpha_1 e^{-\tau}\|\Delta f\|\right) \\ &\leq C\,|\eta|\,e^{-2\tau}\left(\|W\|_{H^1}^2 + \|W\|_{H^1}\|W\|_{H^2}\right)\left(\|f\| + \alpha_1 e^{-\tau}\|\Delta f\|\right) \\ &\leq CM^2\gamma^2\,(1-\theta)^{12}\,e^{-\frac{3\tau}{2}}. \end{aligned}$$

Thus, assuming $\gamma \leq 1$, the following inequality holds:

$$I_4 \leq CM^2\gamma\,(1-\theta)^6\,e^{-\tau}. \tag{4.51}$$

It remains to estimate I_5, which is the term that does not appear in the second grade fluids equations. We rewrite

$$I_5 = I_5^1 + I_5^2,$$

where

$$\begin{aligned} I_5^1 &= -\beta e^{-2\tau} \left(\operatorname{div} \left(\nabla \left(|A|^2\right) \wedge A\right), |X|^4 f\right)_{L^2}, \\ I_5^2 &= \beta \alpha_1 e^{-3\tau} \left(\operatorname{div} \left(\nabla \left(|A|^2\right) \wedge A\right), |X|^4 \Delta f\right)_{L^2}. \end{aligned}$$

We begin by estimating I_5^1. After some computations, we notice that we have to estimate two kinds of terms. In fact, one has

$$I_5^1 \leq I_5^{1,1} + I_5^{1,2},$$

where

$$\begin{aligned} I_5^{1,1} &= C\beta e^{-2\tau} \int_{\mathbb{R}^2} |X|^4 \left|\nabla^2 U\right|^2 |\nabla U| \, |f| \, dX, \\ I_5^{1,2} &= C\beta e^{-2\tau} \int_{\mathbb{R}^2} |X|^4 \left|\nabla^3 U\right| |\nabla U|^2 \, |f| \, dX. \end{aligned}$$

In order to simplify the notations, we define

$$\delta = M\gamma \left(1-\theta\right)^6.$$

Applying the inequality (2.6) of Lemma 2.2 and the continuous injection of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we obtain

$$\begin{aligned} I_5^{1,1} &\leq \beta e^{-2\tau} \left\| |X| \, \nabla^2 U \right\|_{L^4}^2 \|\nabla U\|_{L^\infty} \left\| |X|^2 f \right\| \\ &\leq C\beta e^{-2\tau} \left(\|W\| + \||X| \, \nabla W\|\right) \left(\|\nabla W\| + \||X| \, \Delta W\|\right) \|\nabla U\|_{H^2} \left\| |X|^2 f \right\|. \end{aligned}$$

Then, using the inequalities (2.4) of Lemma 2.2 and (2.17) of Lemma 2.4 and the conditions (4.4) and (4.5), we get

$$\begin{aligned} I_5^{1,1} &\leq C\beta e^{-2\tau} \left(\|W\| + \|\nabla W\|^{1/2} \left\| |X|^2 \nabla W \right\|^{1/2} \right) \\ &\qquad \times \left(\|\nabla W\| + \|\Delta W\|^{1/2} \left\| |X|^2 \Delta W \right\|^{1/2} \right) \|W\|_{H^2} \left\| |X|^2 f \right\| \\ &\leq C\delta e^{-3\tau/2} \left(\delta^{1/2} + \delta^{1/4} \left\| |X|^2 \nabla W \right\|^{1/2} \right) \left(\delta^{1/2} + \delta^{1/4} e^{\tau/4} \left\| |X|^2 \Delta W \right\|^{1/2} \right). \end{aligned}$$

Then, we recall that $W = \eta G + f$. Due to the fact that $|\eta| \leq \delta^{1/2}$ and the smoothness of G, we obtain

$$\begin{aligned} I_5^{1,1} &\leq C\delta e^{-3\tau/2} \left(\delta^{1/2} + \delta^{1/4} \left\| |X|^2 \nabla f \right\|^{1/2} \right) \left(\delta^{1/2} + \delta^{1/4} e^{\tau/4} \left\| |X|^2 \Delta f \right\|^{1/2} \right) \\ &\leq C\delta^2 e^{-3\tau/2} + C\delta^{7/4} e^{-3\tau/2} \left\| |X|^2 \nabla f \right\|^{1/2} + C\delta^{7/4} e^{-5\tau/4} \left\| |X|^2 \Delta f \right\|^{1/2} \\ &\qquad + C\delta^{3/2} e^{-5\tau/4} \left\| |X|^2 \nabla f \right\|^{1/2} \left\| |X|^2 \Delta f \right\|^{1/2}. \end{aligned}$$

Using the Young inequalities $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$ and $ab \leq \frac{1}{3}a^3 + \frac{2}{3}b^{3/2}$, the inequality (4.5) and assuming $\gamma \leq 1$, we finally obtain

$$\begin{aligned}
I_5^{1,1} &\leq C\delta^2 e^{-\frac{3\tau}{2}} + C\delta^2 \left(\left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) \\
&\qquad + C\delta^{5/3} \left(e^{-2\tau} + e^{-\frac{5\tau}{3}} \right) + C\delta^{4/3} e^{-\frac{5\tau}{3}} \left\| |X|^2 \nabla f \right\|^{2/3} \\
&\leq C\delta^2 e^{-\frac{3\tau}{2}} + C\delta^2 \left(\left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right) \\
&\qquad + C\delta^{5/3} \left(e^{-2\tau} + e^{-\frac{5\tau}{3}} \right) + C\delta e^{-\frac{5\tau}{2}} + C\delta^2 \left\| |X|^2 \nabla f \right\|^2 \\
&\leq CM^2\gamma (1-\theta)^6 e^{-\frac{3\tau}{2}} + CM^2\gamma^2 (1-\theta)^{12} \left(\left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right).
\end{aligned} \tag{4.52}$$

In order to estimate $I_5^{1,2}$, we use Hölder inequalities and obtain

$$I_5^{1,2} \leq C\beta e^{-2\tau} \left\| |X|^2 \nabla^3 U \right\| \left\| |X|^2 f \right\|_{L^4} \|\nabla U\|_{L^8}^2 .$$

Then, using the Galiardo-Niremberg inequality, we notice that

$$\begin{aligned}
\left\| |X|^2 f \right\|_{L^4} &\leq \left\| |X|^2 f \right\|^{1/2} \left\| \nabla \left(|X|^2 f \right) \right\|^{1/2} \\
&\leq C \left\| |X|^2 f \right\|^{1/2} \left(\| |X| f \| + \left\| |X|^2 \nabla f \right\| \right)^{1/2} .
\end{aligned}$$

The inequalities (2.13) of lemma 2.3, (2.20) of lemma 2.5 and the continuous injection of $H^1(\mathbb{R}^2)$ into $L^8(\mathbb{R}^2)$ imply

$$\begin{aligned}
I_5^{1,2} &\leq C\beta e^{-2\tau} \left\| |X|^2 \nabla^3 U \right\| \left\| |X|^2 f \right\|_{L^4} \|\nabla U\|_{L^8}^2 \\
&\leq C\beta e^{-2\tau} \left(\|W\| + \| |X| \nabla W \| + \left\| |X|^2 \Delta W \right\| \right) \\
&\qquad \times \left\| |X|^2 f \right\|^{1/2} \left(\| |X| f \| + \left\| |X|^2 \nabla f \right\| \right)^{1/2} \|W\|_{H^1}^2 .
\end{aligned}$$

Finally, using the conditions (4.4) and (4.5) and the Young inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$, we obtain

$$\begin{aligned}
I_5^{1,2} &\leq C\delta^{7/4} e^{-\frac{3\tau}{4}} \left\| |X|^2 f \right\|^{1/2} \\
&\leq C\delta^2 e^{-\tau} + C\delta \left\| |X|^2 f \right\|^2 \\
&\leq CM^2\gamma^2 (1-\theta)^{12} e^{-\tau} + CM\gamma (1-\theta)^6 \left\| |X|^2 f \right\|^2 .
\end{aligned} \tag{4.53}$$

Thus, combining the inequalities (4.52) and (4.53), we obtain

$$I_5^1 \leq CM^2\gamma (1-\theta)^6 e^{-\tau} + CM^2\gamma (1-\theta)^6 \left(\left\| |X|^2 f \right\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right). \tag{4.54}$$

It remains to estimate I_5^2. Like in the case of I_5^1, we have to consider two kinds of terms. Indeed, one can show that

$$I_5^2 \le I_5^{2,1} + I_5^{2,2},$$

where

$$I_5^{2,1} = C\alpha_1\beta e^{-3\tau} \int_{\mathbb{R}^2} |X|^4 \left|\nabla^2 U\right|^2 |\nabla U| \, |\Delta f| \, dX,$$
$$I_5^{2,2} = C\alpha_1\beta e^{-3\tau} \int_{\mathbb{R}^2} |X|^4 \left|\nabla^3 U\right| |\nabla U|^2 \, |\Delta f| \, dX.$$

With the same tools than the ones used to estimate I_5^1, one can bound $I_5^{2,1}$. Due to Hölder inequalities and the continuous injection of $H^2(\mathbb{R}^3)$ into $L^\infty(\mathbb{R}^2)$, one has

$$\begin{aligned} I_5^{2,1} &\le C\alpha_1\beta e^{-3\tau} \left\||X|^2 \Delta f\right\| \left\||X| \nabla^2 U\right\|_{L^4}^2 \|\nabla U\|_{L^\infty} \\ &\le C\alpha_1\beta e^{-3\tau} \left\||X|^2 \Delta f\right\| \left\||X| \nabla^2 U\right\|_{L^4}^2 \|\nabla U\|_{H^2}. \end{aligned}$$

Then, using the inequality (2.6) of Lemma 2.2 and the inequality (2.17) of Lemma 2.4, we obtain

$$I_5^{2,1} \le C\alpha_1\beta e^{-3\tau} \left\||X|^2 \Delta f\right\| (\|W\| + \||X| \nabla W\|) (\|\nabla W\| + \||X| \Delta W\|) \|W\|_{H^2}$$

Finally, the condition (4.4) and Young inequality imply

$$\begin{aligned} I_5^{2,1} &\le C\delta^{3/2} e^{-\tau} \left\||X|^2 \Delta f\right\| \\ &\le CM^2\gamma^2 (1-\theta)^{12} e^{-2\tau} + CM\gamma (1-\theta)^6 \left\||X|^2 \Delta f\right\|^2. \end{aligned} \tag{4.55}$$

Likewise, using the inequality (2.20) of Lemma 2.5 and the continuous injection of $H^{\frac{3}{2}}(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned} I_5^{2,2} &\le C\beta\alpha_1 e^{-3\tau} \left\||X|^2 \Delta f\right\| \left\||X|^2 \nabla^3 U\right\| \|\nabla U\|_{L^\infty}^2 \\ &\le C\beta\alpha_1 e^{-3\tau} \left\||X|^2 \Delta f\right\| \left(\|W\| + \||X| \nabla W\| + \left\||X|^2 \Delta W\right\|\right) \|W\|_{H^{3/2}}^2 \\ &\le C\delta^{1/2} e^{-2\tau} \left\||X|^2 \Delta f\right\| \|W\|_{H^{3/2}}^2. \end{aligned}$$

Using the well-known interpolation inequality

$$\|v\|_{H^{3/2}} \le C \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2}, \qquad \text{for every } v \in H^2(\mathbb{R}^2),$$

we obtain, using again the condition (4.4) and Young inequality,

$$\begin{aligned} I_5^{2,2} &\le C\delta^{1/2} e^{-2\tau} \left\||X|^2 \Delta f\right\| \|W\|_{H^1} \|W\|_{H^2} \\ &\le C\delta^{3/2} e^{-3\tau/2} \left\||X|^2 \Delta f\right\| \\ &\le CM^2\gamma^2 (1-\theta)^{12} e^{-3\tau} + CM\gamma (1-\theta)^6 \left\||X|^2 \Delta f\right\|^2. \end{aligned} \tag{4.56}$$

Finally, the inequalities (4.55) and (4.56) imply

$$I_5^2 \leq CM^2\gamma^2 (1-\theta)^{12} e^{-3\tau} + CM\gamma (1-\theta)^6 \left\| |X|^2 \Delta f \right\|^2 . \tag{4.57}$$

Thus, combining the inequalities (4.54) and (4.57), we get

$$I_5 \leq CM^2\gamma (1-\theta)^6 e^{-\tau} + CM^2\gamma (1-\theta)^6 \left(\left\| |X|^2 f \right\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right). \tag{4.58}$$

Taking into account the inequalities (4.49), (4.50), (4.51) and (4.58) and going back to (4.48), one has

$$\begin{aligned}
\partial_\tau E_6 + \frac{1}{2} \left\| |X|^2 f \right\|^2 + \left(1 + \alpha_1 e^{-\tau}\right) \left\| |X|^2 \nabla f \right\|^2 + \left(\alpha_1 e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau} \right) \left\| |X|^2 \Delta f \right\|^2 \\
- \left(8 + 8\alpha_1 e^{-\tau}\right) \left\| |X| f \right\|^2 \leq \\
C\varepsilon e^{-\tau} \left\| |X| \nabla f \right\|^2 + C\varepsilon \alpha_1 e^{-2\tau} \left\| |X| \Delta f \right\|^2 + CM^2\gamma (1-\theta)^6 e^{-\tau} \\
+ CM^2\gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 \right) \\
+ CM^2\gamma^{1/2} (1-\theta)^3 \left(\left\| |X|^2 f \right\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right).
\end{aligned} \tag{4.59}$$

Via the Young inequality and the condition (4.5), it is easy to check that

$$\begin{aligned}
C\varepsilon e^{-\tau} \left\| |X| \nabla f \right\|^2 + C\varepsilon \alpha_1 e^{-2\tau} \left\| |X| \Delta f \right\|^2 &\leq \varepsilon^2 \left\| |X|^2 \nabla f \right\|^2 + Ce^{-2\tau} \|\nabla f\|^2 \\
&\quad + \varepsilon^2 \left\| |X|^2 \Delta f \right\|^2 + C\alpha_1^2 e^{-4\tau} \|\Delta f\|^2 \\
&\leq \varepsilon^2 \left\| |X|^2 \nabla f \right\|^2 + \varepsilon^2 \left\| |X|^2 \Delta f \right\|^2 \\
&\quad + CM (1-\theta)^6 e^{-2\tau}.
\end{aligned}$$

We assume that $\varepsilon^2 \leq \min \left(\frac{1}{2}, \frac{\alpha_1 e^{-\tau_0}}{2} \right)$. The inequality (4.59) becomes

$$\begin{aligned}
\partial_\tau E_6 + \frac{1}{2} \left\| |X|^2 f \right\|^2 + \left(\frac{1}{2} + \alpha_1 e^{-\tau} \right) \left\| |X|^2 \nabla f \right\|^2 \\
+ \left(\frac{\alpha_1}{2} e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau} \right) \left\| |X|^2 \Delta f \right\|^2 - 8\alpha_1 e^{-\tau} \left\| |X| f \right\|^2 \leq \\
CM^2\gamma (1-\theta)^6 e^{-\tau} + 8 \left\| |X| f \right\|^2 \\
+ C_1 M^2\gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 \right) \\
+ C_1 M^2\gamma^{1/2} (1-\theta)^3 \left(\left\| |X|^2 f \right\|^2 + \left\| |X|^2 \nabla f \right\|^2 + \left\| |X|^2 \Delta f \right\|^2 \right),
\end{aligned} \tag{4.60}$$

where C_1 is a positive constant dependent on α_1 and β.

We take now γ sufficiently small so that $C_1 M^2 \gamma^{1/2} (1-\theta)^3 \leq \frac{1-\theta}{4}$. We obtain

$$\begin{aligned}\partial_\tau E_6 + \left(\frac{\theta}{2} + \frac{1-\theta}{4}\right) \left\||X|^2 f\right\|^2 + \left(\frac{1}{4} + \alpha_1 e^{-\tau}\right) \left\||X|^2 \nabla f\right\|^2 \\ + \left(\frac{\alpha_1}{4} e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau}\right) \left\||X|^2 \Delta f\right\|^2 - 8\alpha_1 e^{-\tau} \left\||X| f\right\|^2 \leq \\ CM^2 \gamma (1-\theta)^6 e^{-\tau} + 8 \left\||X| f\right\|^2 \\ + C_1 M^2 \gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2\right). \end{aligned} \tag{4.61}$$

Using the inequality (2.4) of lemma 2.2, one has

$$8 \left\||X| f\right\|^2 \leq h \left\||X|^2 f\right\|^2 + \frac{64}{h} \|f\|^2, \quad \text{for all} \quad h > 0.$$

Thus, we set $h = \frac{1-\theta}{8}$ and obtain

$$\begin{aligned}\partial_\tau E_6 + \left(\frac{\theta}{2} + \frac{1-\theta}{8}\right) \left\||X|^2 f\right\|^2 + \left(\frac{1}{4} + \alpha_1 e^{-\tau}\right) \left\||X|^2 \nabla f\right\|^2 \\ + \left(\frac{\alpha_1}{4} e^{-\tau} + \frac{\alpha_1^2}{2} e^{-2\tau}\right) \left\||X|^2 \Delta f\right\|^2 - 8\alpha_1 e^{-\tau} \left\||X| f\right\|^2 \leq \\ CM^2 \gamma (1-\theta)^6 e^{-\tau} + \frac{1024}{1-\theta} \|f\|^2 \\ + C_1 M^2 \gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2\right). \end{aligned} \tag{4.62}$$

Integrating several times by parts, we notice that

$$E_6 = \tfrac{1}{2} \left\||X|^2 f\right\|^2 + \alpha_1 e^{-\tau} \left\||X|^2 \nabla f\right\|^2 + \tfrac{\alpha_1^2}{2} e^{-2\tau} \left\||X|^2 \Delta f\right\|^2 - 8\alpha_1 e^{-\tau} \left\||X| f\right\|^2.$$

Consequently, the inequality (4.62) can be written

$$\begin{aligned}\partial_\tau E_6 + \theta E_6 + \frac{1-\theta}{8} \left\||X|^2 f\right\|^2 + \frac{1}{4} \left\||X|^2 \nabla f\right\|^2 + \frac{\alpha_1}{4} e^{-\tau} \left\||X|^2 \Delta f\right\|^2 \\ \leq CM^2 \gamma (1-\theta)^6 e^{-\tau} + \frac{1024}{1-\theta} \|f\|^2 + CM^2 \gamma^{1/2} (1-\theta)^3 \left(\|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2\right). \end{aligned} \tag{4.63}$$

□

5 Proof of Theorem 1.1

In this section, we consider the solution W_ε of (4.1) with initial data W_0 satisfying the condition (1.8) for some $\gamma > 0$ and we take advantage of the energy estimates obtained in Section 4 to show that W_ε satisfies the inequality (1.9). Then, we pass to the limit when ε tends to 0 and show that W_ε converges, up to a subsequence, to a weak solution of (1.6) which satisfies also the inequality (1.9). We recall that

$$W_\varepsilon = \eta G + f_\varepsilon,$$

where G is the Oseen vortex sheet given by (1.3), $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(X)dX$ and f_ε satisfies the equality (4.3). We define the functional

$$E_7 = \frac{K}{1-\theta} E_5 + E_6,$$

where K is a large positive constant that will be made more precise later and E_5 and E_6 are the energy functionals defined in Section 4.

If K is large enough, this energy is suitable to estimate the $H^2(2)$ norm of f_ε, as it is shown by the next lemma.

Lemma 5.1 *Let $f_\varepsilon \in C^1\left((\tau_0,\tau_\varepsilon), H^1(2)\right) \cap C^0\left((\tau_0,\tau_\varepsilon), H^3(2)\right)$. If K is large enough, there exist two positive constants C_1 and C_2 such that, for all $\tau \in (\tau_0,\tau_\varepsilon)$,*

$$E_7 \le \frac{C_1}{1-\theta}\left(\|f_\varepsilon\|_{H^1}^2 + \alpha_1 e^{-\tau}\|\Delta f_\varepsilon\|^2 + \left\||X|^2 f_\varepsilon\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f_\varepsilon\right\|^2\right),$$
$$C_2\left(\|f_\varepsilon\|_{H^1}^2 + \alpha_1 e^{-\tau}\|\Delta f_\varepsilon\|^2 + \left\||X|^2 f_\varepsilon\right\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f_\varepsilon\right\|^2\right) \le E_7.$$

Proof: The first inequality of this lemma comes directly from the definition of E_7. To prove the second one, we notice that

$$E_7 \ge \frac{CK}{1-\theta}\left(\|f_\varepsilon\|_{H^1}^2 + \alpha_1 e^{-\tau}\|\Delta f_\varepsilon\|^2\right) + \frac{1}{2}\left\||X|^2\left(f_\varepsilon - \alpha_1 e^{-\tau}\Delta f_\varepsilon\right)\right\|^2.$$

Furthermore, we have already shown that

$$\begin{aligned}\left\||X|^2\left(f_\varepsilon - \alpha_1 e^{-\tau}\Delta f_\varepsilon\right)\right\|^2 = &\left\||X|^2 f_\varepsilon\right\|^2 + 2\alpha_1 e^{-\tau}\left\||X|^2 \nabla f_\varepsilon\right\|^2 \\ &+\alpha_1^2 e^{-2\tau}\left\||X|^2 \Delta f_\varepsilon\right\|^2 - 16\left\||X| f_\varepsilon\right\|^2.\end{aligned}$$

Via the Hölder and Young inequalities, we get

$$\left\||X|^2\left(f_\varepsilon-\alpha_1 e^{-\tau}\Delta f_\varepsilon\right)\right\|^2 \geq \left\||X|^2 f_\varepsilon\right\|^2+2\alpha_1 e^{-\tau}\left\||X|^2\nabla f_\varepsilon\right\|^2 +\alpha_1^2 e^{-2\tau}\left\||X|^2\Delta f_\varepsilon\right\|^2-\frac{1}{2}\left\||X|^2 f_\varepsilon\right\|^2-128\left\|f_\varepsilon\right\|^2.$$

Consequently, one has

$$E_7 \geq \frac{CK}{1-\theta}\left(\|f_\varepsilon\|_{H^1}^2+\alpha_1 e^{-\tau}\|\Delta f_\varepsilon\|^2\right)+\frac{1}{4}\left\||X|^2 f_\varepsilon\right\|^2+\alpha_1 e^{-\tau}\left\||X|^2\nabla f_\varepsilon\right\|^2 +\frac{\alpha_1^2}{2}e^{-2\tau}\left\||X|^2\Delta f\right\|^2-64\left\|f_\varepsilon\right\|^2.$$

Thus, if K is big enough, we get the second inequality of this lemma.

□

Lemma 5.2 *Let $W_\varepsilon \in C^0\left(\left[\tau_0,\tau_\varepsilon\right), H^3(2)\right)$ be a solution of (4.1) satisfying the inequality (4.4) for some $\gamma>0$. There exist $T_0>0$ and $\gamma_0>0$ such that if $T=e^{\tau_0}\geq T_0$ and $\gamma\leq\gamma_0$, then, for all $\tau\in[\tau_0,\tau_\varepsilon^*)$, E_7 satisfies the inequality*

$$\partial_\tau E_7+\theta E_7 \leq CM^3\gamma\left(1-\theta\right)e^{-\tau}. \tag{5.1}$$

Proof: We take γ_0 and T_0 respectively as small and large as necessary to satisfy the conditions of the lemmas 4.2 to 4.6. According to the inequalities (4.41) and (4.42), one has

$$\begin{aligned}\partial_\tau E_7+\theta E_7+\frac{K}{1-\theta}\left(7\left\|(-\Delta)^{-\frac{1+\theta}{4}}f_\varepsilon\right\|^2+\frac{1}{4}\|\nabla f_\varepsilon\|^2+\frac{1}{4}\|\Delta f_\varepsilon\|^2\right)\\ +\frac{1-\theta}{8}\left\||X|^2 f_\varepsilon\right\|^2+\frac{\alpha_1}{4}e^{-\tau}\left\||X|^2\Delta f_\varepsilon\right\|^2 \leq CM^3\gamma\left(1-\theta\right)e^{-\tau}+\frac{1024}{1-\theta}\|f_\varepsilon\|^2\\ +CM^2\left(1-\theta\right)\gamma^{1/2}K\left(\left\||X|^2 f_\varepsilon\right\|^2+\alpha_1^2 e^{-2\tau}\left\||X|^2\Delta f_\varepsilon\right\|^2\right)\\ +CM^2\left(1-\theta\right)\gamma^{1/2}\left(\|f_\varepsilon\|^2+\|\nabla f_\varepsilon\|^2+\|\Delta f_\varepsilon\|^2\right).\end{aligned}$$

Using the interpolation inequality (4.27) of $\|f_\varepsilon\|^2$ between $\left\|(-\Delta)^{-\frac{1+\theta}{4}}f_\varepsilon\right\|^2$ and $\|\nabla f_\varepsilon\|^2$ and taking K large enough and γ small enough, we get

$$\partial_\tau E_7+\theta E_7 \leq CM^3\gamma\left(1-\theta\right)e^{-\tau}. \tag{5.2}$$

□

Remark 5.1 *We can see in the proofs of the lemmas 4.2 to 5.2 that γ_0 does not depend on θ, but only on α_1, β and M.*

5.1 Regularized problem

Before proving Theorem 1.1, we show an intermediate theorem. This one gives the same result than Theorem 1.1, but for the solution of the regularized system (4.1).

Theorem 5.1 *Let θ be a constant such that $0 < \theta \leq 1$. There exist $\varepsilon_0 = \varepsilon_0(\alpha_1, \beta) > 0$, $\gamma_0 = \gamma_0(\alpha_1, \beta) > 0$ and $T_0 = T_0(\alpha_1, \beta) \geq 0$ such that, for all $\varepsilon \leq \varepsilon_0$, $T = e^{\tau_0} \geq T_0$ and $W_0 \in H^2(2)$ satisfying the condition (1.8) with $\gamma \leq \gamma_0$, there exist a unique global solution $W_\varepsilon \in C^1\left((\tau_0, +\infty), H^1(2)\right) \cap C^0\left((\tau_0, +\infty), H^3(2)\right)$ of (4.1) and a positive constant $C = C(\alpha_1, \beta, \theta) > 0$ such that, for all $\tau \geq \tau_0$,*

$$\left\|\left(1 - \alpha_1 e^{-\tau}\Delta\right)\left(W_\varepsilon(\tau) - \eta G\right)\right\|^2_{L^2(2)} \leq C\gamma e^{-\theta\tau}, \tag{5.3}$$

where $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(x)dx$ and the parameters α_1 and β are fixed and given in (1.1).

Proof of Theorem 5.1: Let $W_0 \in H^2(2)$ satisfying the condition (1.8) with $0 \leq \gamma \leq \gamma_0$ and $0 \leq T_0 \leq T$, where γ_0 and T_0 will be made more precise later. By theorem 3.1, there exist $\tau_\varepsilon > \tau_0 = \log(T)$ and a solution W_ε to the system (4.1) which belongs to $C^1\left((\tau_0, \tau_\varepsilon), H^1(2)\right) \cap C^0\left((\tau_0, \tau_\varepsilon), H^3(2)\right)$. Let $\eta = \displaystyle\int_{\mathbb{R}^2} W_0(X)dX$, and f_ε defined by the equality

$$W_\varepsilon = \eta G + f_\varepsilon. \tag{5.4}$$

Let $M > 2$ be a positive constant that will be set later and $\tau^*_\varepsilon \in [\tau_0, \tau_\varepsilon)$ be the highest positive time such that the inequality (4.4) holds. As shown at the beginning of Section 4, the inequality (4.5) holds on $[\tau_0, \tau^*_\varepsilon)$. We take T_0 sufficiently large and γ_0 and ε sufficiently small so that the results of the lemmas 4.2 to 5.2 occur. Consequently, there exists $C = C(\alpha_1, \beta) > 0$ such that, for all $\tau \in [\tau_0, \tau^*_\varepsilon)$,

$$\partial_\tau\left(E_7 e^{\theta\tau}\right) \leq CM^3\gamma\left(1 - \theta\right)e^{-(1-\theta)\tau}. \tag{5.5}$$

Integrating this inequality in time between τ_0 and $\tau \in [\tau_0, \tau^*_\varepsilon)$, we obtain

$$E_7(\tau) \leq E_7(\tau_0)e^{-\theta(\tau-\tau_0)} + CM^3\gamma\left(e^{-(1-\theta)\tau_0}e^{-\theta\tau} - e^{-\tau}\right). \tag{5.6}$$

Due to the decomposition (5.4) and the lemma 5.1, for every $\tau \in [\tau_0, \tau^*_\varepsilon)$, one has

$$\|W_\varepsilon(\tau)\|^2_{H^1} + \left\||X|^2 W_\varepsilon(\tau)\right\|^2 + \alpha_1 e^{-\tau}\|\Delta W_\varepsilon(\tau)\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2\Delta W_\varepsilon(\tau)\right\|^2 \leq C\eta^2 + CE_7(\tau).$$

Since f_ε satisfies the inequality (4.5), one has $\eta^2 \leq C\gamma\,(1-\theta)^6$. Taking into account the inequality (5.6), it comes

$$\|W_\varepsilon(\tau)\|_{H^1}^2 + \left\||X|^2\, W_\varepsilon(\tau)\right\|^2 + \alpha_1 e^{-\tau}\,\|\Delta W_\varepsilon(\tau)\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2\,\Delta W_\varepsilon(\tau)\right\|^2 \leq C\gamma\,(1-\theta)^6 + E_7(\tau_0)e^{-\theta(\tau-\tau_0)} + CM^3\gamma e^{-\tau_0}. \tag{5.7}$$

Using again the lemma 5.1 and arguing like for the establishment of the inequality (4.5), we can show that

$$\begin{aligned} E_7(\tau_0) \quad &\leq \frac{C}{1-\theta}\Big(\,\|f_\varepsilon(\tau_0)\|_{H^1}^2 + \alpha_1 e^{-\tau_0}\,\|\Delta f_\varepsilon(\tau_0)\|^2 \\ &\qquad + \left\||X|^2\, f_\varepsilon(\tau_0)\right\|^2 + \alpha_1^2 e^{-2\tau_0}\left\||X|^2\,\Delta f_\varepsilon(\tau_0)\right\|^2\Big) \\ &\leq C\gamma\,(1-\theta)^5\,. \end{aligned}$$

Consequently, the inequality (5.7) becomes

$$\|W_\varepsilon(\tau)\|_{H^1}^2 + \left\||X|^2\, W_\varepsilon(\tau)\right\|^2 + \alpha_1 e^{-\tau}\,\|\Delta W_\varepsilon(\tau)\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2\,\Delta W_\varepsilon(\tau)\right\|^2 \leq C_1\gamma\,(1-\theta)^5 + C_2 M^3\gamma e^{-\tau_0}, \tag{5.8}$$

where C_1 and C_2 are two positive constants independent of W_0 and θ.

We set $M = \frac{4C_1}{1-\theta}$, and we get

$$\|W_\varepsilon(\tau)\|_{H^1}^2 + \left\||X|^2\, W_\varepsilon(\tau)\right\|^2 + \alpha_1 e^{-\tau}\,\|\Delta W_\varepsilon(\tau)\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2\,\Delta W_\varepsilon(\tau)\right\|^2 \leq \frac{M\gamma\,(1-\theta)^6}{4} + C_2 M^3\gamma e^{-\tau_0}. \tag{5.9}$$

Finally, taking T_0 sufficiently large so that $C_2 M^3\gamma e^{-\tau_0} \leq \dfrac{M\gamma\,(1-\theta)^6}{4}$, we obtain, for all $\tau \in [\tau_0, \tau_\varepsilon^*)$,

$$\|W_\varepsilon(\tau)\|_{H^1}^2 + \left\||X|^2\, W_\varepsilon(\tau)\right\|^2 + \alpha_1 e^{-\tau}\,\|\Delta W_\varepsilon(\tau)\|^2 + \alpha_1^2 e^{-2\tau}\left\||X|^2\,\Delta W_\varepsilon(\tau)\right\|^2 \leq \frac{M\gamma\,(1-\theta)^6}{2}. \tag{5.10}$$

This inequality shows in particular that $\tau_\varepsilon^* = \tau_\varepsilon$ and thus (5.10) holds for all $\tau \in [\tau_0, \tau_\varepsilon)$. From the inequality (5.10), we deduce also that $\tau_\varepsilon = +\infty$. Indeed, if $\tau_\varepsilon < +\infty$, the boundedness of W_ε in $H^2(2)$ on $[\tau_0, \tau_\varepsilon)$ given by (5.10) is a contradiction to the finiteness of τ_ε.

In particular, the inequality (5.6) occurs on $[\tau_0, +\infty)$. Applying the lemma 5.1 in the inequality (5.6), we finally obtain the inequality (5.3).

5.2 Existence of weak solutions in $H^2(2)$

Now, we show that under the hypotheses of Theorem 5.1, there exists a global weak solution W of (1.6) which belongs to $C^0([\tau_0,+\infty), H^2(2))$, and that this solution converges to the Oseen vortex sheet G when τ goes to infinity. To this end, we pass to the limit in the system (4.1) when ε tends to 0 and show that, up to a subsequence, W_ε converges in some sense to a solution of the system (1.6) which satisfies the inequality (5.3). Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to 0. We consider the solution $W_{\varepsilon_n} \in C^1((\tau_0,+\infty), H^1(2)) \cap C^0((\tau_0,+\infty), H^3(2))$ of (4.1) which satisfies the conditions of Theorem 5.1. Due to technical reasons linked to the compactness properties of Sobolev spaces, it is more convenient to establish the convergence of W_{ε_n} to W in every regular bounded domain of $\mathbb{R}^2$. Let Ω be a regular domain of $\mathbb{R}^2$ and τ_1 be a fixed positive time such that $\tau_0 < \tau_1 < +\infty$. In what follows, $H^s(\Omega)$, $s \geq 0$, denotes the restrictions to Ω of the functions of the Sobolev space $H^s(\mathbb{R}^2)$. From Theorem 5.1, we know that W_{ε_n} is bounded in $L^\infty([\tau_0,+\infty), H^2(2))$ uniformly with respect to n. Consequently, there exists $W \in L^\infty([\tau_0,\tau_1], H^2(2))$ such that

$$W_{\varepsilon_n} \rightharpoonup W \quad \text{weakly in} \quad L^p([\tau_0,\tau_1], H^2(\Omega)), \text{ for all } p \geq 2.$$

Looking at the system (4.1), we can see that $\partial_\tau W_{\varepsilon_n}$ is bounded in $L^\infty([\tau_0,\tau_1], H^1(\Omega))$ uniformly with respect to n. This implies that W_{ε_n} is equicontinuous in $H^1(\Omega)$. Indeed, for $\sigma_1, \sigma_2 \in [\tau_0,\tau_1]$, $\sigma_2 \geq \sigma_1$, we have

$$\begin{aligned}\|W_{\varepsilon_n}(\sigma_2) - W_{\varepsilon_n}(\sigma_1)\|_{H^1(\Omega)} &= \left\| \int_{\sigma_1}^{\sigma_2} \partial_\tau W_{\varepsilon_n}(s)ds \right\|_{H^1(\Omega)} \\ &\leq (\sigma_2 - \sigma_1) \|\partial_\tau W_{\varepsilon_n}(s)\|_{L^\infty([\tau_0,\tau_1],H^1(\Omega))}.\end{aligned}$$

Furthermore, for every $\tau \in [\tau_0,\tau_1]$, the set $\bigcup_{n\in\mathbb{N}} f_{\epsilon_n}(\tau)$ is bounded in $H^2(\Omega)$ and thus compact in $H^1(\Omega)$. Using the Arzela-Ascoli theorem, we get

$$W_{\varepsilon_n} \to W \quad \text{strongly in} \quad C^0([\tau_0,\tau_1], H^1(\Omega)).$$

By interpolation, we can show that

$$W_{\varepsilon_n} \to W \quad \text{in} \quad C^0([\tau_0,\tau_1], H^s(\Omega)), \quad \text{for all} \quad s < 2. \tag{5.11}$$

This is enough to pass to the limit in the system (4.1) in the sense of distributions on $[\tau_0,\tau_1] \times \Omega$ and to show that W is a weak solution of the system (1.6). Since most of the terms of the equation (4.1) have already been studied in [47], we will just show that the convergence holds for the term $-\text{div curl}\,\left(|A_{\varepsilon_n}|^2 A_{\varepsilon_n}\right)$ which does not appear in the second grade fluids equations.

We consider $\varphi \in C_0^\infty([\tau_0,\tau_1]\times\Omega)$. For all $\tau \in [\tau_0,\tau_1]$, we want to show that

$$\begin{aligned} &\int_{\tau_0}^{\tau}\int_\Omega |A_{\varepsilon_n}(\tau,X)|^2 A_{\varepsilon_n}(\tau,X)\diamond\nabla^2\varphi(\tau,X)dX\,d\tau \longrightarrow \\ &\qquad\qquad \int_{\tau_0}^{\tau}\int_\Omega |A(\tau,X)|^2 A(\tau,X)\diamond\nabla^2\varphi(\tau,X)dX\,d\tau, \end{aligned} \tag{5.12}$$

when n tends to infinity, where, for $A,B\in\mathcal{M}_2(\mathbb{R})$, we use the notation

$$A\diamond B=\sum_{j=1}^{2}\left(A_{1,j}B_{2,j}-A_{2,j}B_{1,j}\right).$$

The term of the right hand side of (5.12) appears naturally via two integrations by parts, when performing the L^2-scalar product of $-\text{div curl}\,\left(|A|^2 A\right)$ with φ. The strong convergence of W_{ε_n} to W in $C^0\left([\tau_0,\tau_1],H^1(\Omega)\right)$ implies directly the identity (5.12). Indeed, due to the continuous injection of $H^1(\Omega)$ into $L^3(\Omega)$, W_{ε_n} converges to W in $C^0\left([\tau_0,\tau_1],L^3(\Omega)\right)$. Furthermore, the inequality (2.13) implies

$$\|A_{\varepsilon_n}-A\|_{L^3}\le\|W_{\varepsilon_n}-W\|_{L^3},$$

and consequently A_{ε_n} converges to A strongly in $C^0\left([\tau_0,\tau_1],L^3(\Omega)\right)$. This fact suffices to show that the identity (5.12) occurs. Thus W is a global weak solution of (1.6) which belongs to $C^0\left([\tau_0,+\infty),H^2(2)\right)$.

The fact that W satisfies the inequality (1.9) is a direct consequence of the weak convergence of W_{ε_n} to W. Indeed, for all $\tau\in[\tau_0,+\infty)$, $W_{\varepsilon_n}(\tau)$ is bounded in $H^2(2)$ uniformly with respect to n and consequently we have

$$W_{\varepsilon_n}(\tau)\rightharpoonup W(\tau),\text{ weakly in } H^2(2),\text{ for all } \tau\in[\tau_0,+\infty).$$

Since W_{ε_n} satisfies the inequality (1.9), it implies that W also satisfies (1.9).

5.3 Uniqueness

The aim of this part is to prove that the solution w of the system (1.2) obtained in Section 5.2 is unique in $L^2(2)$. Let w_1 and w_2 be two solutions of (1.2) with the same initial data $w_0 \in H^2(2)$. Let u_1 and u_2 be the divergence free vector fields obtained via the Biot-Savart law respectively from w_1 and w_2. We also define $A_i = \nabla u_i + (\nabla u_i)^t$. Applying the Biot-Savart law to the system (1.1), we can see that, for $i = 1, 2$, the divergence free vector field u_i satisfies the system

$$\begin{aligned}&\partial_t (u_i - \alpha_1 \Delta u_i) - \Delta u_i + \text{curl } (u_i - \alpha_1 \Delta u_i) \wedge u_i - \beta \text{div } \left(|A_i|^2 A_i\right) + \nabla p_i = 0,\\ &\text{div } u_i = 0,\\ &u_{i|t=0} = u_0,\end{aligned} \tag{5.13}$$

where u_0 is obtained from w_0 via the Biot-Savart law.

Notice that since w_i belongs to $L^\infty_{loc}(\mathbb{R}^+, H^2(2))$ and $\partial_t w_i$ belongs to $L^\infty_{loc}(\mathbb{R}^+, H^1(\mathbb{R}^2))$, the inequalities (2.11) and (2.13) imply in particular

$$\begin{aligned}&u_i \in L^\infty_{loc}(\mathbb{R}^+, L^p(\mathbb{R}^2)^2), \quad \text{for all} \quad p > 2,\\ &\nabla u_i \in L^\infty_{loc}(\mathbb{R}^+, H^2(\mathbb{R}^2)^4),\\ &\partial_t u_i \in L^\infty_{loc}(\mathbb{R}^+, L^p(\mathbb{R}^2)^2), \quad \text{for all} \quad p > 2,\\ &\partial_t \Delta u_i \in L^\infty_{loc}(\mathbb{R}^+, L^2(\mathbb{R}^2)^2).\end{aligned}$$

Consequently, the system (5.13) has a meaning in the sense of distributions.

We note $w = w_1 - w_2$, $u = u_1 - u_2$, $L = L_1 - L_2$ and $A = A_1 - A_2$. A short computation shows that u satisfies the system

$$\begin{aligned}&\partial_t (u - \alpha_1 \Delta u) - \Delta u + \text{curl } (u - \alpha_1 \Delta u) \wedge u_1 + \text{curl } (u_2 - \alpha_1 \Delta u_2) \wedge u\\ &\qquad\qquad + \beta \text{div } \left(|A_2|^2 A_2\right) - \beta \text{div } \left(|A_1|^2 A_1\right) + \nabla q = 0,\\ &\text{div } u = 0,\\ &u_{|t=0} = 0.\end{aligned} \tag{5.14}$$

Notice that, although u_1 and u_2 do not belong to $L^2(\mathbb{R}^2)$, the divergence free vector field u does. Indeed, since w_1 and w_2 have the same initial data, for all $t \geq 0$, we have

$$\int_{\mathbb{R}^2} w(t,x)dx = 0.$$

By application of the lemma 2.5, this fact implies that u belongs to $L^2(\mathbb{R}^2)$. Let $t_0 > 0$ be a fixed positive time. We notice that both w_1 and w_2 are bounded in $L^\infty([0, t_0], H^2(2))$. More precisely, one has

$$\sup_{t\in[0,t_0]}\Big(\|w_1(t)\|_{H^2(2)}+\|w_2(t)\|_{H^2(2)}\Big)\leq C.$$

Applying the lemma 2.3, it implies in particular

$$\sup_{t\in[0,t_0]}\left(\|u_i(t)\|_{L^4}+\|\nabla u_i(t)\|_{L\infty}+\|\Delta u_i(t)\|_{L^4}\right)\leq C,\quad \text{for}\quad i=1,2.$$

In order to show that $u\equiv 0$, we now perform estimates on the H^1-norm of u. The uniqueness of the solutions of (5.13) has been shown in [10] for solutions with initial data in $H^2(\mathbb{R}^2)$. In our case, the proof is slightly simpler, because the vector field u belongs to $H^3(\mathbb{R}^2)^2$. We consider the L^2-inner product of (5.14) with u. First of all, integrating by parts, we notice that

$$\begin{aligned}\beta\left(\operatorname{div}\left(|A_2|^2A_2-|A_1|^2A_1\right),u\right)_{L^2}&=\frac{\beta}{2}\left(|A_1|^2A_1-|A_2|^2A_2,A\right)_{L^2}\\&=\frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A_1|^2+|A_2|^2\right)|A|^2\,dx\\&\qquad+\frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A_1|^2-|A_2|^2\right)(A_1+A_2):A dx\\&=\frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A_1|^2+|A_2|^2\right)|A|^2\,dx\\&\qquad\qquad+\frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A_1|^2-|A_2|^2\right)^2dx.\end{aligned}$$

Thus, using integrations by parts and the divergence free property of u, we have

$$\begin{aligned}\frac{1}{2}\partial_t\left(\|u\|^2+\alpha\|\nabla u\|^2\right)+\|\nabla u\|^2+\frac{\beta}{4}\int_{\mathbb{R}^2}&\left(|A_1|^2+|A_2|^2\right)|A|\,dx\\&+\frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A_1|^2-|A_2|^2\right)^2dx=I_1+I_2,\end{aligned}\tag{5.15}$$

where

$$\begin{aligned}I_1&=\left(\operatorname{curl}\,(u_2-\alpha_1\Delta u_2)\wedge u,u\right)_{L^2},\\I_2&=\left(\operatorname{curl}u\wedge u_1,u\right)_{L^2},\\I_3&=-\alpha_1\left(\operatorname{curl}\Delta u\wedge u_1,u\right)_{L^2}.\end{aligned}$$

A short computation shows that I_1 vanishes. Indeed, we set $\omega=u_2-\alpha_1\Delta u_2$ and we recall the notation $u=(u^1,u^2,0)$ and $\operatorname{curl}\omega=(0,0,\partial_1\omega_2-\partial_2\omega_1)$. We have

$$\begin{aligned}I_1&=\left(\operatorname{curl}\omega\wedge u,u\right)_{L^2}\\&=-\Big(\left(\partial_1\omega_2-\partial_2\omega_1\right)u^2,u^1\Big)_{L^2}+\Big(\left(\partial_1\omega_2-\partial_2\omega_1\right)u^1,u^2\Big)_{L^2}\\&=0.\end{aligned}$$

Due to the boundedness of u_1 in $L^4(\mathbb{R}^2)$, applying Hölder inequalities we obtain

$$\begin{aligned} I_2 &\leq \|u_1\|_{L^4} \|\nabla u\| \|u\| \\ &\leq C(\alpha_1) \left(\|u\|^2 + \alpha_1 \|\nabla u\|^2 \right). \end{aligned}$$

Using [57, Lemma A.1], we check that

$$I_3 \leq C\alpha_1 \int_{\mathbb{R}^2} |\Delta u_1| |\nabla u| |u| \, dx + C\alpha_1 \int_{\mathbb{R}^2} |\nabla u_1| |\nabla u|^2 \, dx.$$

Using Hölder inequalities, the Gagliardo-Nirenberg inequality and the Young inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$, we obtain

$$\begin{aligned} I_3 &\leq C\alpha_1 \|u\|_{L^4} \|\Delta u_2\|_{L^4} \|\nabla u\| + C\alpha_1 \|\nabla u_1\|_{L^\infty} \|\nabla u\|^2 \\ &\leq C\alpha_1 \|\nabla u\|^{3/2} \|u\|^{1/2} + C\alpha_1 \|\nabla u\|^2 \\ &\leq C(\alpha_1) \left(\|u\|^2 + \alpha_1 \|\nabla u\|^2 \right). \end{aligned}$$

Going back to (5.15), we get

$$\frac{1}{2}\partial_t \left(\|u\|^2 + \alpha \|\nabla u\|^2 \right) \leq C(\alpha) \left(\|u\|^2 + \alpha \|\nabla u\|^2 \right). \tag{5.16}$$

Integrating in time this inequality between 0 and $t \in [0, t_0]$ and applying the Gronwall lemma, we finally obtain

$$\|u(t)\|^2 + \alpha \|\nabla u(t)\|^2 = 0, \quad \text{for all} \quad t \in [0, t_0].$$

Since t_0 is arbitrary, we conclude that $u \equiv 0$ on $\mathbb{R}^+$. Consequently u is unique and so is w. Thus, the system (1.2) has a unique global solution in the space $C^0(\mathbb{R}^+, H^2(2))$.

II. Attractor for the third grade fluids equations in dimension 2

1 Introduction

In this article, we study the asymptotic behaviour of solutions of third grade fluids equations in the periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$. This class of non Newtonian fluids, which is a particular case of fluids of grade n (or Rivlin Ericksen fluids), has been introduced from the mathematical point of view by Fosdick and Rajagopal in 1980 (see [31] and [59]). Since one can find many non Newtonian fluids in the nature, the understanding of their behaviours is important. For instance, one can find non-Newtonian fluids in a lot of oils used in industry, or even in the day-life, for examples melted cheese or wet sand. The third grade fluids equations are given by

$$\begin{aligned}
&\partial_t (u - \alpha_1 \Delta u) - \nu \Delta u + \text{curl } (u - \alpha_1 \Delta u) \wedge u - (\alpha_1 + \alpha_2) (A \Delta u + \text{div } (L L^t)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad -\beta \text{div } \left(|A|^2 A \right) + \nabla p = f, \\
&\text{div } u = 0, \\
&u_{|t=0} = u_0,
\end{aligned} \tag{1.1}$$

where u is a vector field of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\nu \geq 0$ is the cinematic viscosity, p is the pressure of the fluid, depending on u, $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, $\beta \geq 0$, $(L)_{i,j} = \partial_j u_i$ and $A = L + L^t$. For matrices $A, B \in \mathcal{M}_d(\mathbb{R})$, we use the notation $|A|^2 = \sum_{i,j=1}^{d} A_{i,j}^2$ and $A : B = \sum_{i,j=1}^{d} A_{i,j} B_{i,j}$.

Notice that if we consider $\beta = 0$ and $\alpha_2 = -\alpha_1$, one recovers the second grade fluids equations, which is another class of non-Newtonian fluids, introduced earlier in 1974 by

J. Dunn and R. Fosdick (see [24]). If we assume $\alpha_1 = \alpha_2 = \beta = 0$, one recovers the classical Navier-Stokes equations, which modelize Newtonian fluids.

The equations of third grade fluids have been studied in various cases, in open domains of $\mathbb{R}^2$ or $\mathbb{R}^3$ (see [1], [9], [5] or [61]) or in the whole spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ (see [10], [11] or [55]). On a bounded set of $\mathbb{R}^2$ or $\mathbb{R}^3$, Amrouche and Cioranescu have shown the existence and uniqueness of local solutions to (1.1) in the Sobolev space H^3 with Dirichlet boundary conditions (see [1]). For this study, they made the restriction

$$|\alpha_1 + \alpha_2| \leq \sqrt{24\nu\beta}, \tag{1.2}$$

which comes from physical considerations. The proof of their result is based on a Galerkin method associated to the eigenspaces of the operator curl $(Id - \alpha\Delta)$. Later, in [9], Bresch and Lemoine introduced a more general class of solutions belonging to the space $L^\infty\left([0,T], W^{2,p}(\Omega)^3\right)$, with $p > 3$ and $T > 0$, for initial data in $W^{2,p}(\Omega)^3$ and forcing term in $L^p\left(\mathbb{R}^+, L^p(\Omega)^3\right)$. This result is obtained without assumption on the parameters by using the Schauder's fixed point theorem, and shows additionally that the solutions are global in time if the initial data are small enough in $W^{2,p}(\Omega)^3$. In the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$, Busuioc and Iftimie established the existence of global weak solutions in the Sobolev space H^2 (see [10]), using Friedrich's method and a priori estimates in H^2. Furthermore, they showed the uniqueness property in dimension 2 and the propagation of the regularity if the data are in H^3. Later, Paicu established the existence of global weak solutions in the Sobolev space H^1 (see [55]), considering additional restrictions on the parameters α_1, α_2 and β. The methods used in [55] are slightly different from the ones used in [10]. The proof of Paicu involves also a Friedrich's scheme, but the final step which shows that the approximate solutions converge to a solution of (1.1) is achieved through a monotonicity method, that we will use in the present paper. Moreover, it is shown that these weak solutions satisfy an energy equality, which will be useful in the present paper.

In this paper, we are interested in the asymptotic behaviour of weak H^1-solutions of third grade fluids equations with periodic conditions. In order to study these asymptotics, we will associate a dynamical system $T(t)$ to the solutions of (1.1), as it has been made for the second grade fluids equations or the Navier-Stokes equations. On a bounded set Ω of $\mathbb{R}^2$ with homogeneous boundary conditions, Ladyzhenskaya has shown that the solutions of the Navier-Stokes equations with initial data in the functions space $H = \left\{u \in L^2(\Omega)^2 : \operatorname{div} u = 0, u_{|\partial\Omega} = 0\right\}$ define a dynamical system in H (see [49] and

[50]). The same result has been shown by Moise, Rosa and Wang for the second grade fluids equations on the space $H^3 \cap H$ (see [53]). In our case, the situation is slightly more complicated. Indeed, since the solutions of (1.1) are not known to be unique in H^1, we cannot associate a classical dynamical system to the H^1-weak solutions. To overcome this difficulty, we show that the set of the weak H^1-solutions of (1.1) is a generalized semiflow for the H^1-topology, according to the definition of Ball (see [3] and [4]). In [3, Theorem 3.3], J. Ball gives a useful theorem that shows the existence of a compact global attractor, provided that the generalized semiflow is point-dissipative and asymptotically compact. This result is the analogue of a theorem shown by Hale, Lasalle and Slemrod in [44] for classical dynamical systems (see also [43]). For the second grade fluids equations, the existence of a compact global attractor has been shown by Moire, Rosa and Wang on a bounded or periodic domain of $\mathbb{R}^2$ for solutions with initial data in H^3, assuming that the forcing term is constant in time and belongs to H^1 (see [53]). This result is obtained through an energy equality method, which show that the dynamical system associated to the solutions of the second grade fluids equations is point dissipative and asymptotically compact. In the case of the torus of dimension 2, Paicu, Rekalo and Raugel proved that this attractor belongs to a more regular space than H^3 (see [57]). Indeed, they proved that there exists $\delta > 0$ such that if the forcing term is in $H^{1+\delta}$, then the attractor is bounded in $H^{3+\delta}$. They also extended the regularity results to the case of solutions with initial data in $W^{3,p}(\mathbb{T}^2)$, where $1 < p < +\infty$ and $W^{s,q}$ denotes the Sobolev space of order s associated to the L^q-norm (see [56]). To obtain this result, they considered Lagrangian coordinates.

In this paper, using the works of Paicu [55] and Busuioc and Iftimie [10], we will show that the weak solutions of (1.1) in dimension 2 with periodic boundary conditions and initial data in H^2 admit a compact global attractor for the H^1-topology. This result is obviously weaker than the ones obtained for the second grade fluids equations, and is mainly due to the fact that, under restrictions on the parameters α_1, ν and β, the H^2-solutions of (1.1) admit a bounded absorbing set in H^2.

In the two dimensional case, combining results that we can find in [10] and [1], one can show that the system (1.1) is equivalent to

$$\begin{aligned}
&\partial_t (u - \alpha\Delta u) - \nu\Delta u + u.\nabla u - \alpha \mathrm{div}\ (u.\nabla A + L^t A + AL) - \beta \mathrm{div}\ \left(|A|^2 A\right) + \nabla p = f,\\
&\mathrm{div}\ u = 0,\\
&u_{|t=0} = u_0,
\end{aligned} \tag{1.3}$$

where $\alpha = \alpha_1$.

Notice that, in this new system, the constant α_2 disappeared. This phenomenon is due to the divergence free property of u and is very particular to the dimension 2. The proofs of the results of this paper clearly take advantage of this fact.

The plan of this paper is as follows. In the section 2, we establish the existence of weak solutions to (1.3) with initial data in $H^1(\mathbb{T}^2)^2$ and forcing term in $L^2(\mathbb{T}^2)^2$ which is independent of the time. We will also recall the energy equality satisfied by these solutions, which has a particular form in dimension 2. In the section 3, we recall some definitions about the generalized semiflows and then show that the set of the weak solutions of (1.3) with initial data in H^1 is a generalized semiflow for the H^1-topology. Furthermore, we show that this generalized semiflow admits a bounded absorbing set. Finally, we show in the section 4 that the set of the weak solutions of (1.3) with initial data in H^2 admits in some sense a compact global attractor for the H^1-topology. More precisely, we will show that the bounded sets of H^2 are attracted in the topology of H^1 by an invariant bounded set of H^2.

2 Existence of weak solutions and energy equality

In this section, we show that the proof of the existence of global H^1 solutions to (1.3) given by Paicu in [55] for the whole space $\mathbb{R}^2$ extends to the case of $\mathbb{T}^2$. To this end, we define the functions spaces

$$H = \left\{ u \in L^2(\mathbb{T}^2)^2 : \operatorname{div} u = 0, \int_{\mathbb{T}^2} u(x)dx = 0 \right\}, \quad \text{and} \quad V = H^1(\mathbb{T}^2)^2 \cap H.$$

In particular, every $u \in V$ satisfies the Poincaré inequality

$$\|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 . \tag{2.1}$$

One defines also, for $s > 0$, the functions space

$$V^s = H^s(\mathbb{T}^2)^2 \cap H,$$

and V^{-s}, the dual space of V^s.

Like for the case of a fluid filling the whole space $\mathbb{R}^2$, we show next that, given initial data in V, there exist weak solutions to (1.3) belonging to the space

$$X_\infty = L^\infty\left(\mathbb{R}^+, V\right) \cap L^4_{loc}\left(\mathbb{R}^+, W^{1,4}\left(\mathbb{T}^2\right)^2\right).$$

Although this result has not been written, it is almost shown in [55], and only a few details differ between the periodic and the whole space cases.

In order to define the weak solutions of (1.3), we introduce the space

$$X_T = L^\infty\left([0,T], V\right) \cap L^4\left([0,T], W^{1,4}(\mathbb{T}^2)\right),$$

its dual space X_T' and the operator $R : X_T \to X_T'$ given by

$$R(u) = -\nu\Delta u - \alpha \mathrm{div}\ \left(L^t A + AL\right) - \beta \mathrm{div}\ \left(|A|^2 A\right).$$

This non-linear operator is continuous from X_T to X_T' and

$$\langle R(u), v\rangle_{X_T', X_T} = \int_0^T \Bigg[\nu\left(\nabla u(s), \nabla v(s)\right)_{L^2} + \alpha\left(\left(L^t A + AL\right)(s), \nabla v(s)\right)_{L^2} + \beta\left(|A(s)|^2 A(s), \nabla v(s)\right)_{L^2}\Bigg] ds.$$

Furthermore, assuming that the parameter α is small compared to the product $\nu\beta$, we have the following monotonicity result.

Lemma 2.1 *Assume that $|\alpha| \leq \sqrt{8\nu\beta}$. Then R is monotone, that is*

$$\langle R(u) - R(v), u - v\rangle_{X_T', X_T} \geq 0, \tag{2.2}$$

for all $u, v \in X_T$.

The proof of this lemma is given in [12].

We now give the definition of a weak solution of (1.3).

Definition 2.1 *Let $u_0 \in V$ and $f \in H$.*

We say that $u \in C^0\left(\mathbb{R}^+, V\right) \cap L^4_{loc}\left(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2\right)$ is a weak solution of (1.3) with initial datum u_0 if, for all $T > 0$ and $\varphi \in C^1\left([0,T], V^2\right)$, the following equality holds:

$$\begin{aligned}
&\left(u(T), \varphi(T) - \alpha\Delta\varphi(T)\right)_{L^2} + \langle R(u), \varphi\rangle_{X_T', X_T} + \int_0^T \left(u(s).\nabla u(s), \varphi(s)\right)_{L^2} ds \\
&\qquad - \alpha \sum_{i,j,k=1}^2 \int_0^T \int_{\mathbb{T}^2} u_k(s) A^{i,j}(s) \partial_k \partial_j \varphi_i(s) dx ds \\
&= \left(u_0, \varphi(0) - \alpha\Delta\varphi(0)\right)_{L^2} + \int_0^T \left(u(s), \partial_t\left(\varphi(s) - \alpha\Delta\varphi(s)\right)\right)_{L^2} ds + \int_0^T \left(f, \varphi(s)\right)_{L^2} ds.
\end{aligned} \tag{2.3}$$

Remark 2.1 *If u is a weak solutions of (1.3), then the equality*

$$\partial_t \left(u - \alpha \Delta u\right) - \nu \Delta u + u.\nabla u - \alpha div \left(L^t A + AL + u.\nabla A\right) - \beta div \left(\left|A\right|^2 A\right) + \nabla p = f \quad (2.4)$$

occurs in the sense of distributions.

For later use, we define an equivalent norm to the classical H^1-norm, that is

$$\|u\|_{H^1_\alpha}^2 = \|u\|_{L^2}^2 + \alpha \|\nabla u\|_{L^2}^2 .$$

We establish now a weak existence result for the system (1.3), when (u_0, f) belongs to $V \times L^2(\mathbb{T}^2)^2$. This result comes nearly directly from the existence result of Paicu in the whole space $\mathbb{R}^2$, but there are some details to adapt to the case of periodic boundary conditions.

Theorem 2.1 *Assume that $\nu \geq 0$, $\beta > 0$ and $|\alpha| \leq \sqrt{8\nu\beta}$ and let $u_0 \in V$ and $f \in H$ be given. There exists a solution u to the system (1.3) such that*

$$u \in C_b^0 \left(\mathbb{R}^+, V\right) \cap L^4_{loc} \left(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)\right) \quad \text{and} \quad \partial_t u \in L^\infty_{loc} \left(\mathbb{R}^+, H\right).$$

In addition, for every weak solution of (1.3) in the sense of Definition 2.3 and all $t \geq s \geq 0$, the following energy equality holds

$$\frac{1}{2} \|u(t)\|_{H^1_\alpha}^2 + \nu \int_s^t \|\nabla u(\sigma)\|_{L^2}^2 \, d\sigma + \frac{\beta}{2} \int_s^t \|A(\sigma)\|_{L^4}^4 \, d\sigma = \frac{1}{2} \|u(s)\|_{H^1_\alpha}^2 + \int_s^t \left(f, u(\sigma)\right)_{L^2} d\sigma. \quad (2.5)$$

The proof of this theorem is very similar to the one in [55]. We only need to do a few changes. In particular, we need a technical lemma, which is the analogue the lemma obtained in [23, lemma II.1] on the whole space $\mathbb{R}^2$. For $n \in \mathbb{N}$ and v a function of $\mathbb{R}^2$, we define the linear operator

$$J_n(v) = \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}^2} \varphi\left(\frac{|k|}{2^n}\right) \widehat{v}_k e^{ik.x},$$

where $\widehat{v}_k = \displaystyle\int_{\mathbb{T}^2} v(x) e^{-ik.x} dx$ is the classical Fourier coefficient of v associated to k, and $\varphi \in C_0^\infty \left(\mathbb{R}, [0,1]\right)$ is a smooth function such that

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq \frac{3}{4}, \\ 0, & |\xi| \geq 1. \end{cases}$$

It is well-known that, for all $p \geq 2$, J_n is well defined on $L^p(\mathbb{T}^2)$ and, for all $v \in L^p(\mathbb{T}^2)$, one has

$$J_n(v) \underset{n\to\infty}{\longrightarrow} v, \text{ in } L^p(\mathbb{T}^2).$$

Furthermore, J_n is self-adjoint with respect to the classical L^2-scalar product and commutes with derivatives

The lemma which enables to prove Theorem 2.1 is the following.

Lemma 2.2 *1) Let $u \in W^{1,4}(\mathbb{T}^2)^2$ and $v \in L^4(\mathbb{T}^2)$, then one has*

$$J_n(u.\nabla v) - u.\nabla J_n(v) \to 0 \text{ in } L^2(\mathbb{T}^2) \text{ when } n \to +\infty.$$

2) Let $T \geq 0$, $u \in L^4\left([0,T], W^{1,4}(\mathbb{T}^2)^2\right)$ and $v \in L^4\left([0,T]\, L^4(\mathbb{T}^2)\right)$, then one has

$$J_n(u.\nabla v) - u.\nabla J_n(v) \to 0 \text{ in } L^2\left([0,T], L^2(\mathbb{T}^2)\right) \text{ when } n \to +\infty.$$

This lemma is obtained by following step by step the proof of [23, lemma II.1] given by R. Di Perna and P. -L. Lions and enables to show the existence of weak solutions of (1.3), following the steps of [55, Theorem 1]. The proof of the energy equality is also given in [55]. Indeed, it is shown that, for all $t \geq s \geq 0$ and every weak solution u, one has

$$\begin{aligned} &\frac{1}{2}\|u(t)\|^2_{H^1_\alpha} + \nu \int_s^t \|\nabla u(\sigma)\|^2_{L^2}\, d\sigma + \frac{\beta}{2}\int_s^t \|A(\sigma)\|^4_{L^4}\, d\sigma \\ &\qquad + \frac{1}{2}\alpha \int_s^t \int_{\mathbb{T}^2} \operatorname{tr}\left(A^3(\sigma)\right) dx d\sigma = \frac{1}{2}\|u(s)\|^2_{H^1_\alpha} + \int_s^t (f, u(\sigma))_{L^2}\,. \end{aligned} \tag{2.6}$$

Actually, using the divergence free property of u, one obtains

$$\begin{aligned} \operatorname{tr}\left(A^3\right) &= \sum_{i,j,k=1}^{2} A^{i,k}A^{k,j}A^{i,j} \\ &= A^{1,1}A^{1,1}A^{1,1} + A^{2,2}A^{2,2}A^{2,2} + 3A^{1,2}A^{1,2}A^{1,1} + 3A^{1,2}A^{1,2}A^{2,2} \\ &= 0. \end{aligned}$$

Thus, in dimension 2, the energy equality (2.5) holds. This phenomenon is very particular to the dimension 2 and does not occur in dimension 3.

Remark 2.2 *Notice that the fact that u belongs to $L^4_{loc}\left(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2\right)$ is a consequence of the fact that u belongs to $L^\infty_{loc}\left(\mathbb{R}^+, V\right)$ and A belongs to $L^4_{loc}\left(\mathbb{R}^+, L^4(\mathbb{T}^2)^2\right)$. According to the Korn inequality and the continuous injection of $H^1(\mathbb{T}^2)$ into $L^4(\mathbb{T}^2)$, it implies that u belongs to $L^4_{loc}\left(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2\right)$.*

3 Generalized semiflow

Since the weak solutions of (1.3) are not known to be unique in general, there are new difficulties which do not occur for the second grade fluids equations or the Navier-Stokes equations. In fact, we cannot associate a classical dynamical system to the weak solutions of (1.3). However, we will see that these solutions define a generalized semiflow on V, according to the definition of J. Ball (see [3] or [4]). Furthermore, using again the results of Ball, we are able to show the existence of a compact attractor in V for weak solutions of (1.3) with initial data in a bounded set of V^2. In this section, we will also assume additional restrictions on the parameters α, β and ν. Hereafter, we assume that

$$|\alpha| \leq \sqrt{8\nu\beta},$$

so that Lemma 2.2 and Theorem 2.1 hold.

First of all, we recall some definitions about generalized semiflows and their asymptotic behaviour. In what follows, X denotes a metric space.

Definition 3.1 *A generalized semiflow G on X is a family of maps $\varphi : [0, +\infty) \to X$ (called solutions) satisfying the hypotheses:*

(H1) (Existence) For each $z \in X$ there exists at least one $\varphi \in G$ with $\varphi(0) = z$.

(H2) (Translates of solutions are solutions) If $\varphi \in G$ and $\tau \geq 0$, then $\varphi^\tau \in G$, where $\varphi^\tau(t) = \varphi(t+\tau)$, $\quad t \in [0, +\infty)$.

(H3) (Concatenation) If $\varphi, \psi \in G$, $t \geq 0$, with $\psi(0) = \varphi(t)$ then $\theta \in G$, where

$$\theta(\tau) = \begin{cases} \varphi(\tau), & \text{for} \quad 0 \leq \tau \leq t, \\ \psi(\tau - t), & \text{for} \quad t < \tau. \end{cases}$$

(H4) (Upper-semicontinuity with respect to initial data) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

We will see later that the set of the solutions of (1.3) is a generalized semiflow on V. For a generalized semiflow G, we state the definition below.

Definition 3.2 *Let G be a generalized semiflow on X. A complete orbit is a map $\varphi : \mathbb{R} \to X$ such that for any $s \in \mathbb{R}$, $\varphi^s \in G$. If φ is a complete orbit, we define the α-limit of φ, given by*

$$\alpha(\varphi) = \left\{ z \in X : \text{ there exists a sequence } t_j \to -\infty \text{ such that } \varphi(t_j) \underset{j\to\infty}{\longrightarrow} z \right\}.$$

Let $E \subset X$, the ω-limit set of E is the set given by

$$\omega(E) = \Big\{ z \in X : \text{ there exist } \varphi_j \in G \text{ with } \varphi_j(0) \in E, \varphi_j(0) \text{ bounded,} \\ \text{and a sequence } t_j \in \mathbb{R}^+, t_j \to +\infty, \text{ such that } \varphi_j(t_j) \underset{j\to\infty}{\longrightarrow} z \Big\}.$$

If G is a generalized semiflow on a metric space (X, d), we note 2^X the space of all the subsets of X and we define the application $T(t) : 2^X \to 2^X$ such that, for $E \subset X$,

$$T(t)E = \{\varphi(t) : \varphi \in G, \varphi(0) \in E\}. \tag{3.1}$$

We also define the semi-distance δ_X on the subsets of X, given by, for $E, F \subset X$,

$$\delta_X(E, F) = \sup_{e\in E} \inf_{f\in F} d(e, f).$$

Notice that δ_X is not symmetric, that is the reason why it is not a distance on 2^X.

Definition 3.3 *Let G be a generalized semiflow on X.*

1. *G is point-dissipative if there exists a bounded set $B \subset X$ such that, for all $\varphi \in G$, there exists $T \geq 0$ such that $\varphi(t) \in B$, for all $t \geq T$.*
2. *G is asymptotically compact if for any sequence $\varphi_j \in G$ such that $\varphi_j(0)$ is bounded, and any sequence t_j such that, $t_j \to +\infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.*
3. *The subset $A \subset X$ attracts the subset $B \subset X$ if $\delta_X(T(t)B, A) \to 0$ when $t \to +\infty$.*
4. *The subset $A \subset X$ is positively invariant if $T(t)A \subset A$, for all $t \geq 0$.*
5. *The subset $A \subset X$ is quasi-invariant if, for each $z \in A$, there exists a complete orbit $\varphi \in G$ such that $\varphi(0) = z$ and $\varphi(t) \in A$, for all $t \in \mathbb{R}$.*
6. *The subset $A \subset X$ is invariant if $T(t)A = A$, for all $t \geq 0$.*
7. *The subset $A \subset X$ is a compact global attractor if A is compact, invariant and attracts the bounded sets of X.*

For later use, we also introduce the following definition.

Definition 3.4 *Let G be a generalized semiflow on a metric space X. We say that $B \subset X$ is a bounded absorbing set of G if B is bounded and, for all bounded set $E \subset X$, there exists $T \equiv T(E) \geq 0$ such that,*

$$T(t)E \subset X, \text{ for all } t \geq T.$$

In particular, every generalized semiflow which admits a bounded absorbing set is point dissipative. We can now state the main result of this section, which shows that the set of the weak solutions of (1.3) is a generalized semiflow on V. Using the energy equality (2.5), one can also show that this generalized semiflow is point-dissipative. In what follows, $\mathcal{W}$ denotes the set of weak solutions of (1.3) in V, that is,

$$\mathcal{W} = \{u \in C^0(\mathbb{R}^+, V) \cap L^4_{loc}(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2) : u \text{ is a weak solution of } (1.3)\}.$$

The main theorem of this section is the following.

Theorem 3.1 *Assume that $\nu > 0$, $\beta > 0$, $\alpha > 0$ and $\alpha < \sqrt{8\nu\beta}$. The set $\mathcal{W}$ of the weak solutions of (1.3) is a generalized semiflow on V, which admits a bounded absorbing set.*

Proof: The property $(H1)$ has already been shown in Theorem 2.1. The properties $(H2)$ and $(H3)$ are easy to check by using the definition of weak solutions. It remains to show the property $(H4)$. Let $u_{0,j}$ be a sequence of V such that $u_{0,j} \to u_0$ strongly in V. We note B the positive constant such that

$$\|u_{0,j}\|^2_{H^1_\alpha} \le B.$$

Let $u_j \in C^0_b(\mathbb{R}^+, V) \cap L^4_{loc}(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2)$ be the weak solutions of (1.3) with initial data $u_{0,j}$. Due to the energy equality (2.5), one has, for all $t \in \mathbb{R}^+$ and $j \in \mathbb{N}$,

$$\|u_j(t)\|^2_{H^1_\alpha} + 2\nu \int_0^t \|\nabla u_j(s)\|^2_{L^2}\, ds + \beta \int_0^t \|A_j(s)\|^4_{L^4}\, ds = \|u_{0,j}\|^2_{H^1_\alpha} + 2\int_0^t (f, u_j(s))_{L^2}\, ds, \tag{3.2}$$

where $A_j = A(u_j)$.

Using Cauchy-Schwartz and Poincaré inequalities, we check that

$$\begin{aligned} 2\int_0^t (f, u_j(s))_{L^2}\, ds &\le C\int_0^t \|f\|_{L^2} \|\nabla u_j(s)\|_{L^2}\, ds \\ &\le \nu \int_0^t \|\nabla u_j(s)\|^2_{L^2}\, ds + \frac{Ct}{\nu}\|f\|^2_{L^2}, \end{aligned}$$

where C is a positive constant independent of the parameters.

Going back to (3.2), we get

$$\|u_j(t)\|^2_{H^1_\alpha} + \nu \int_0^t \|\nabla u_j(s)\|^2_{L^2}\, ds + \beta \int_0^t \|A_j(s)\|^4_{L^4}\, ds \le B + \frac{Ct}{\nu}\|f\|^2_{L^2}. \tag{3.3}$$

Thus, u_j is bounded in $L^\infty_{loc}(\mathbb{R}^+, V) \cap L^4_{loc}(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2)$ uniformly in j. Consequently, there exists $u \in L^\infty_{loc}(\mathbb{R}^+, V) \cap L^4_{loc}(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2)$ and a subsequence of u_j (that we still note u_j) such that

$$\begin{array}{lll} u_j \rightharpoonup u & \text{weak* in} & L^\infty_{loc}(\mathbb{R}^+, V)\,, \\ u_j \rightharpoonup u & \text{weakly in} & L^4_{loc}(\mathbb{R}^+, W^{1,4}(\mathbb{T}^2)^2)\,. \end{array} \tag{3.4}$$

As explained in the remark 2.2, the last identity of (3.4) comes from Korn's inequality and the continuous injection of $H^1(\mathbb{T}^2)$ into $L^4(\mathbb{T}^2)$. Furthermore, using the equality (2.4), we can check that $\partial_t u_j$ is bounded in $L^\infty_{loc}(\mathbb{R}^+, H)$, uniformly with respect to j. This property implies that u_j is equi-continuous in H. Indeed, one has, for all $0 \leq t_1 \leq t_2$,

$$\begin{aligned} \|u_j(t_2) - u_j(t_1)\|_{L^2} &\leq \left\| \int_{t_1}^{t_2} \partial_t u_j(t) dt \right\|_{L^2} \\ &\leq \int_{t_1}^{t_2} \|\partial_t u_j(t)\|_{L^2} \, dt \\ &\leq \|\partial_t u_j\|_{L^\infty([t_1,t_2],L^2)} \, |t_2 - t_1|\,. \end{aligned}$$

Furthermore, for all $t \in \mathbb{R}^+$, the set $\bigcup_{j \in \mathbb{N}} u_j(t)$ is bounded in V and thus compact in H. Using the classical Arzela-Ascoli theorem, we conclude that, for all fixed $T \in \mathbb{R}^+$,

$$u_j \to u \quad \text{strongly in} \quad C^0([0,T], H)\,.$$

Consequently, the boundedness of $u_j(t)$ in V uniformly with respect to j obtained from (3.3) and the strong convergence of u_j to u in $C^0([0,T], H)$ for all $T \geq 0$ imply, by a density argument,

$$u_j(t) \rightharpoonup u(t) \quad \text{weakly in} \quad V, \quad \text{for all} \quad t \in \mathbb{R}^+. \tag{3.5}$$

Besides, using the boundedness property of u_j in $L^\infty([0,T], V)$ and interpolation inequalities, it is easy to check that, for all $0 \leq s < 1$ and $T \in \mathbb{R}^+$,

$$u_j \to u \quad \text{strongly in} \quad C^0([0,T], V^s)\,. \tag{3.6}$$

We show now that u is a weak solution of (1.3). The proof of this point is obtained by following the proof of Paicu to establish theorem 2.1, involving a monotonicity method.

Let T be a fixed positive time and $\varphi \in C^1([0,T], V^2)$. For all $j \in \mathbb{N}$, one has

$$\begin{aligned}(u_j(T), \varphi(T) - \alpha\Delta\varphi(T))_{L^2} + \langle R(u_j), \varphi\rangle_{X'_T, X_T} + \int_0^T (u_j(s).\nabla u_j(s), \varphi(s))_{L^2}\, ds \\ -\alpha \sum_{i,k,l=1}^{2} \int_0^T \int_{\mathbb{T}^2} u_{j,k}(s) A_j^{i,l}(s) \partial_k \partial_l \varphi_i(s) dx ds \\ = (u_{0,j}, \varphi(0) - \alpha\Delta\varphi(0))_{L^2} + \int_0^T (u_j(s), \partial_t(\varphi(s) - \alpha\Delta\varphi(s)))_{L^2}\, ds \\ + \int_0^T (f, \varphi(s))_{L^2}\, ds.\end{aligned} \tag{3.7}$$

Due to the identities (3.4), (3.5) and (3.6), it is quite easy to show that most of the terms of the previous equality pass to the limit when j tends to infinity. The hard term that we have to study more precisely is the term involving the operator R.

Due to the boundedness of u_j in $L^\infty([0,T], V) \cap L^4([0,T], W^{1,4}(\mathbb{T}^2)^2)$, one can check that $R(u_j)$ is bounded in X'_T uniformly in j. Consequently, there exists $\xi \in X'_T$ such that

$$R(u_j) \rightharpoonup \xi \text{ weak* in } X'_T,$$

and thus, for all $\varphi \in C^1([0,T], V^2)$, one has

$$\begin{aligned}(u(T), \varphi(T) - \alpha\Delta\varphi(T))_{L^2} + \langle \xi, \varphi\rangle_{X'_T, X_T} + \int_0^T (u(s).\nabla u(s), \varphi(s))_{L^2}\, ds \\ -\alpha \sum_{i,k,l=1}^{2} \int_0^T \int_{\mathbb{T}^2} u_k(s) A^{i,l}(s) \partial_k \partial_l \varphi_i(s) dx ds \\ = (u_0, \varphi(0) - \alpha\Delta\varphi(0))_{L^2} + \int_0^T (u_j(s), \partial_t(\varphi(s) - \alpha\Delta\varphi(s)))_{L^2}\, ds \\ + \int_0^T (f, \varphi(s))_{L^2}\, ds.\end{aligned} \tag{3.8}$$

In order to show that u is a weak solution of (1.3), it remains to show that $\xi = R(u)$. We establish this equality via a monotonicity method. Actually, it suffices to show that, for all $\psi \in X_T$, one has

$$\langle \xi - R(\psi), u - \psi\rangle_{X'_T, X_T} \geq 0. \tag{3.9}$$

Indeed, assume that the inequality (3.9) holds. Then, setting $\psi = u + \lambda\Psi$, with $\Psi \in X_T$ and $\lambda > 0$, we obtain, for all $\Psi \in X_T$,

$$\langle \xi - R(u + \lambda\Psi), \Psi \rangle_{X'_T, X_T} \leq 0. \tag{3.10}$$

We use now the continuity of R in X_T and let λ go to 0. We get, for all $\Psi \in X_T$,

$$\langle \xi - R(u), \Psi \rangle_{X'_T, X_T} \leq 0. \tag{3.11}$$

Replacing Ψ by $-\Psi$ in the inequality (3.11), we obtain, for all $\Psi \in X_T$,

$$\langle \xi - R(u), \Psi \rangle_{X'_T, X_T} = 0. \tag{3.12}$$

In order to get (3.9), we write the decomposition

$$\begin{aligned}\langle \xi - R(\psi), u - \psi \rangle_{X'_T, X_T} = {} & \langle R(u_j) - R(\psi), u_j - \psi \rangle_{X'_T, X_T} + \langle R(u_j) - \xi, \psi \rangle_{X'_T, X_T} \\ & + \langle \xi - R(\psi), u_j - u \rangle_{X'_T, X_T} + \langle \xi, u \rangle_{X'_T, X_T} - \langle R(u_j), u_j \rangle_{X'_T, X_T}.\end{aligned} \tag{3.13}$$

Due to Lemma 2.2, the convergence of $R(u_j)$ to ξ in X'_T and the convergence of u_j to u in X_T, it is clear that

$$\begin{aligned}&\langle R(u_j) - R(\psi), u_j - \psi \rangle_{X'_T, X_T} \geq 0, \quad \text{for all} \quad j \in \mathbb{N} \\ &\langle R(u_j) - \xi, \psi \rangle_{X'_T, X_T} \to 0, \quad \text{when} \quad j \to +\infty, \\ &\langle \xi - R(\psi), u_j - u \rangle_{X'_T, X_T} \to 0, \quad \text{when} \quad j \to +\infty.\end{aligned} \tag{3.14}$$

In order to finish to establish the property (3.9), we establish the inequality

$$\liminf_{j \to +\infty} \Big(\langle \xi, u \rangle_{X'_T, X_T} - \langle R(u_j), u_j \rangle_{X'_T, X_T} \Big) \geq 0.$$

To get this property, we show that we can apply the equalities (3.7) and (3.8) to respectively $\varphi = u_j$ and $\varphi = u$. Since u and u_j are not smooth enough to make this operation, we consider $n \in \mathbb{N}$, apply (3.7) to $J_n^2(u_j)$ and (3.8) to $J_n^2(u)$ and pass to the limit when $n \to +\infty$. Applying J_n^2 to the equality (2.4) and recalling that u_j and u belong to $C^0(\mathbb{R}^+, V^s)$ for all $s < 1$, it is quite easy to check that $J_n^2(u)$ and $J^2(u)$ belong to $C^1(\mathbb{R}^+, V^2)$. Since J_n is self-adjoint with respect to the L^2-scalar product, an integration by parts gives

$$\begin{aligned}\frac{1}{2}\|J_n(u(T))\|^2_{H^1_\alpha}+\left\langle \xi, J_n^2(u)\right\rangle_{X'_T,X_T}+\int_0^T\left(J_n\left(u(s).\nabla u(s)\right), J_n(u(s))\right)_{L^2}ds\\ +\alpha\int_0^T\left(J_n\left(u(s).\nabla A(s)\right), \nabla J_n(u(s))\right)_{L^2}ds\\ =\left((J_n(u_0), J_n(u(0)))_{L^2}+\alpha\left(\nabla J_n(u_0), \nabla J_n(u(0))\right)_{L^2}\right)-\frac{1}{2}\|J_n(u(0))\|^2_{H^1_\alpha}\\ +\int_0^T\left(J_n(f), J_n(u(s))\right)_{L^2}ds.\end{aligned}\tag{3.15}$$

Since $u_j(t)\rightharpoonup u(t)$ weakly in V for all $t\in\mathbb{R}^+$ and $u_{j,0}=u_j(0)\to u_0$ in V, it is clear that $u(0)=u_0$. Thus, the equality (3.15) becomes

$$\begin{aligned}\frac{1}{2}\|J_n(u(T))\|^2_{H^1_\alpha}+\left\langle \xi, J_n^2(u)\right\rangle_{X'_T,X_T}+\int_0^T\left(J_n\left(u(s).\nabla u(s)\right), J_n(u(s))\right)_{L^2}ds\\ +\alpha\int_0^T\left(J_n\left(u(s).\nabla A(s)\right), \nabla J_n(u(s))\right)_{L^2}ds\\ =\frac{1}{2}\|J_n(u_0)\|^2_{H^1_\alpha}+\int_0^T\left(J_n(f), J_n(u(s))\right)_{L^2}ds.\end{aligned}\tag{3.16}$$

Due to the convergence of $J_n(u)$ to u in X_T, one has obviously, when n tends to infinity,

$$\begin{aligned}&\|J_n(u(T))\|^2_{H^1_\alpha}\to\|u(T)\|^2_{H^1_\alpha},\\ &\left\langle \xi, J_n^2(u)\right\rangle_{X'_T,X_T}\to\langle\xi,u\rangle_{X'_T,X_T},\\ &\int_0^T\left(J_n\left(u(s).\nabla u(s)\right), J_n(u(s))\right)_{L^2}ds\to\int_0^T\left(u(s).\nabla u(s), u(s)\right)_{L^2}ds=0,\\ &\|J_n(u_0)\|^2_{H^1_\alpha}\to\|u_0\|^2_{H^1_\alpha},\\ &\int_0^T\left(J_n(f), J_n(u(s))\right)_{L^2}ds\to\int_0^T\left(f, u(s)\right)_{L^2}ds.\end{aligned}$$

Lemma 2.2 and an integration by parts imply

$$\begin{aligned}\lim_{n\to+\infty}\alpha\int_0^T\left(J_n\left(u(s).\nabla A(s)\right), \nabla J_n(u(s))\right)_{L^2}ds=\\ \lim_{n\to+\infty}\alpha\int_0^T\left(u(s).\nabla J_n(A(s)), \nabla J_n(u(s))\right)_{L^2}ds=0.\end{aligned}$$

Thus, passing to the limit when n goes to infinity in the equality (3.16), we get

$$\langle \xi, u\rangle_{X_T', X_T} = \frac{1}{2}\left(\|u_0\|^2_{H^1_\alpha} - \|u(T)\|^2_{H^1_\alpha}\right) + \int_0^T (f, u(s))_{L^2}\, ds. \tag{3.17}$$

Applying the same method to the equality (3.7), we obtain, for all $j \in \mathbb{N}$,

$$\langle R(u_j), u_j\rangle_{X_T', X_T} = \frac{1}{2}\left(\|u_{0,j}\|^2_{H^1_\alpha} - \|u_j(T)\|^2_{H^1_\alpha}\right) + \int_0^T (f, u_j(s))_{L^2}\, ds. \tag{3.18}$$

Furthermore, due to the fact that $u_j(T) \rightharpoonup u(T)$ weakly in V, one has

$$\|u(T)\|^2_{H^1_\alpha} \leq \liminf_{j\to+\infty} \|u_j(T)\|^2_{H^1_\alpha}.$$

Consequently, using the fact that $u_j \rightharpoonup u$ weakly in $L^2([0,T], H)$ and making the difference between (3.17) and (3.18), we obtain

$$\liminf_{j\to+\infty}\left(\langle \xi, u\rangle_{X_T', X_T} - \langle R(u_j), u_j\rangle_{X_T', X_T}\right) \geq 0. \tag{3.19}$$

Going back to the decomposition (3.13) and letting j go to infinity, the properties (3.19) and (3.14) imply the inequality (3.9). Consequently, u is a weak solution of (1.3).

It remains to show that, for all $t \in \mathbb{R}^+$, $u_j(t) \to u(t)$ strongly in V. From the inequality (2.5) of Theorem 2.1, we have the energy equality

$$\begin{aligned}\frac{1}{2}\|u_j(t)\|^2_{H^1_\alpha} + \frac{\nu}{2}\int_0^t \|\nabla u_j(s)\|^2_{L^2}\, ds + \frac{\beta}{2}\int_0^t \|A_j(s)\|^4_{L^4}\, ds = \\ \frac{1}{2}\|u_{0,j}\|^2_{H^1_\alpha} + \int_0^t (f, u_j(s))_{L^2}\, ds.\end{aligned} \tag{3.20}$$

Due the several convergences of u_j to u given in (3.4), we conclude that

$$\frac{\nu}{2}\int_0^t \|\nabla u(s)\|^2_{L^2}\, ds \leq \liminf_{j\to+\infty} \frac{\nu}{2}\int_0^t \|\nabla u_j(s)\|^2_{L^2}\, ds,$$

$$\frac{\beta}{2}\int_0^t \|A(s)\|^4_{L^4}\, ds \leq \liminf_{j\to+\infty} \frac{\beta}{2}\int_0^t \|A_j(s)\|^4_{L^4}\, ds.$$

Thus, we pass to the lim sup when j tends to infinity in (3.20). We take into account the strong convergence of u_j to u in $C^0\left([0,t],H\right)$ given in (3.6) and get

$$\begin{aligned}\frac{1}{2}\limsup_{j\to+\infty}\left(\|u_j(t)\|^2_{H^1_\alpha}\right)+\frac{\nu}{2}\int_0^t\|\nabla u(s)\|^2_{L^2}\,ds+\frac{\beta}{2}\int_0^t\|A(s)\|^4_{L^4}\,ds\\ \leq\frac{1}{2}\|u_0\|^2_{H^1_\alpha}+\int_0^t\left(f,u(s)\right)_{L^2}ds.\end{aligned}\tag{3.21}$$

Since u is also a weak solution of (1.3), it satisfies the energy equality (2.5). One has

$$\frac{1}{2}\|u(t)\|^2_{H^1_\alpha}+\frac{\nu}{2}\int_0^t\|\nabla u(s)\|^2_{L^2}\,ds+\frac{\beta}{2}\int_0^t\|A(s)\|^4_{L^4}\,ds=\frac{1}{2}\|u_0\|^2_{H^1_\alpha}+\int_0^t\left(f,u(s)\right)_{L^2}ds.\tag{3.22}$$

The difference between (3.21) and (3.22) implies

$$\limsup_{j\to+\infty}\left(\|u_j(t)\|^2_{H^1_\alpha}\right)\leq\|u(t)\|^2_{H^1_\alpha}.\tag{3.23}$$

Since $u_j(t)\rightharpoonup u(t)$ in V, one has also

$$\|u(t)\|^2_{H^1_\alpha}\leq\liminf_{j\to+\infty}\left(\|u_j(t)\|^2_{H^1_\alpha}\right),$$

and thus $\|u_j(t)\|^2_{H^1_\alpha}\to\|u(t)\|^2_{H^1_\alpha}$ when j tends to infinity. From a classical functional analysis result, it implies that $u_j(t)\to u(t)$ strongly in V.

Bounded absorbing set

Finally, we show that $\mathcal{W}$ admits a bounded absorbing set. This property comes nearly directly from the energy equality (2.5). Let $R>0$ and $u_0\in V$ be such that $\|u_0\|_{H^1_\alpha}\leq R$. Let $u\in C^0\left(\mathbb{R}^+,V\right)\cap L^4_{loc}\left(\mathbb{R}^+,W^{1,4}(\mathbb{T}^2)^2\right)$ be a weak solution of (1.3) with initial datum u_0. For $n\in\mathbb{N}$, we apply J_n to the equality (2.4) and take the L^2-inner product of it with J_nu. Performing integrations by parts, one has

$$\begin{aligned}\frac{1}{2}\partial_t\left(\|J_n(u)\|^2_{H^1_\alpha}\right)+\nu\|\nabla J_n(u)\|^2_{L^2}+\left(J_n\left(u.\nabla u\right),J_n(u)\right)_{L^2}\\ +\frac{\alpha}{2}\left(J_n\left(L^tA+AL\right),J_n(A)\right)_{L^2}+\frac{\alpha}{2}\left(J_n\left(u.\nabla A\right),J_n(A)\right)_{L^2}\\ +\frac{\beta}{2}\int_{\mathbb{T}^2}J_n\left(|A|^2A\right)J_n(A)dx=\left(J_n(f),J_n(u)\right)_{L^2}.\end{aligned}\tag{3.24}$$

Using Cauchy-Schwartz, Poincaré (2.1) and Young inequalities, we obtain

$$(J_n(f), J_n(u))_{L^2} \leq \frac{\nu}{2} \|\nabla J_n(u)\|_{L^2}^2 + \frac{1}{2\nu} \|J_n(f)\|_{L^2}^2 . \tag{3.25}$$

Thus, from the inequality (3.24), using again the inequality (2.1) and setting $M = \dfrac{\nu}{4\max(1,\alpha)}$, we obtain, for all $t \in \mathbb{R}^+$ and $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{2}\partial_t \left(\|J_n(u)\|_{H^1_\alpha}^2 e^{Mt}\right) &+ (J_n(u.\nabla u), J_n(u))_{L^2} e^{Mt} + \frac{\alpha}{2}\left(J_n\left(L^tA + AL\right), J_n(A)\right)_{L^2} e^{Mt} \\ &+ \frac{\alpha}{2}(J_n(u.\nabla A), J_n(A))_{L^2} e^{Mt} + \frac{\beta}{2}\int_{\mathbb{T}^2} J_n\left(|A|^2 A\right) J_n(A) e^{Mt} dx \leq \frac{1}{2\nu}\|J_n(f)\|_{L^2}^2 e^{Mt}.\end{aligned} \tag{3.26}$$

We integrate in time the above inequality between 0 and $t > 0$ and get

$$\begin{aligned}\frac{1}{2}\|J_n(u(t))\|_{H^1_\alpha}^2 &+ \int_0^t (J_n(u(s).\nabla u(s)), J_n(u(s)))_{L^2} e^{-M(t-s)} ds \\ &+ \frac{\alpha}{2}\int_0^t \left(J_n\left(L^tA + AL\right)(s), J_n(A(s))\right)_{L^2} e^{-M(t-s)} ds \\ &+ \frac{\alpha}{2}\int_0^t (J_n(u(s).\nabla A(s)), J_n(A(s)))_{L^2} e^{-M(t-s)} ds \\ &+ \frac{\beta}{2}\int_0^t \int_{\mathbb{T}^2} J_n\left(|A|^2 A\right)(s) J_n(A(s)) e^{-M(t-s)} dx ds \\ &\leq \frac{1}{2}\|J_n(u_0)\|_{H^1_\alpha}^2 e^{-Mt} + \frac{1}{2M\nu}\|J_n(f)\|_{L^2}^2 .\end{aligned} \tag{3.27}$$

Due to the boundedness of u in $C^0([0,t], V) \cap L^4([0,t], W^{1,4}(\mathbb{T}^2)^2)$ and the properties of J_n, we have, when n goes to infinity,

$$\begin{aligned}&\int_0^t (J_n(u(s).\nabla u(s)), J_n(u(s)))_{L^2} e^{-M(t-s)} ds \longrightarrow \int_0^t (u(s).\nabla u(s), u(s))_{L^2} e^{-M(t-s)} ds \\ &\qquad = 0, \\ &\int_0^t \left(J_n\left(L^tA + AL\right)(s), J_n(A(s))\right)_{L^2} e^{-M(t-s)} ds \longrightarrow \int_0^t \int_{\mathbb{T}^2} \mathrm{tr}(A^3) e^{-M(t-s)} dx ds = 0, \\ &\int_0^t \int_{\mathbb{T}^2} J_n\left(|A|^2 A\right)(s) J_n(A(s)) e^{-M(t-s)} dx ds \longrightarrow \int_0^t \|A(s)\|_{L^4}^4 e^{-M(t-s)} ds.\end{aligned}$$

Besides, Lemma 2.2 implies

$$\begin{aligned}&\lim_{n\to+\infty}\int_0^t (J_n(u(s).\nabla A(s)), J_n(A(s)))_{L^2} e^{-M(t-s)} ds = \\ &\qquad \lim_{n\to+\infty}\int_0^t (u(s).\nabla J_n(A(s)), J_n(A(s)))_{L^2} e^{-M(t-s)} ds = 0.\end{aligned}$$

Thus, passing to the limit when n tends to infinity, we obtain from the inequality (3.27), for all $t \in \mathbb{R}^+$,

$$\frac{1}{2}\|u(t)\|^2_{H^1_\alpha} + \int_0^t \|A(s)\|^4_{L^4} e^{-M(t-s)} ds \leq \frac{R^2 e^{-Mt}}{2} + \frac{1}{2M\nu}\|f\|^2_{L^2}. \tag{3.28}$$

Consequently, for all $t \geq T = \max\left(0, -\frac{1}{M}\log\left(\frac{\|f\|^2_{L^2}}{2R^2 M\nu}\right)\right)$, we have

$$\frac{1}{2}\|u(t)\|^2_{H^1_\alpha} + \int_0^t \|A(s)\|^4_{L^4} e^{-M(t-s)} ds \leq \frac{1}{M\nu}\|f\|^2_{L^2}, \tag{3.29}$$

and thus, for all $t \geq T$, $u(t)$ belongs to the ball of center 0 and radius $\frac{1}{M\nu}\|f\|^2_{L^2}$.

$\square$

4 Attractor for V^2 solutions

In this section, we show that every bounded set of V^2 is attracted by a compact invariant set of V in the H^1-topology. To obtain this result, we show that the solutions of (1.3) with initial data in V^2 admit a bounded absorbing set in V^2, provided that the viscosity ν and the parameters α and β satisfy some suitable restrictions. This fact will enable us to show the existence of an invariant bounded set of V^2 that attracts all the bounded sets of V^2 in the H^1-topology. We first recall the theorem of Busuioc and Iftimie [10, Theorem 1], that establish the existence of weak solutions of (1.3) with initial data in V^2. In [10], this result has been proved in the case of the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$, but it still holds in the case of periodic conditions. We state here only the two dimensional case, but the same theorem holds in dimension 3, up to the fact that the solutions in $\mathbb{R}^3$ are not known to be unique.

Theorem 4.1 *Assume that $\nu > 0$, $\alpha > 0$ and $\beta > 0$. Let $u_0 \in V^2$ and $f \in H$. There exists a unique global solution u to the system (1.3) such that*

$$u \in L^\infty_{loc}(\mathbb{R}^+, V^2) \cap C^0(\mathbb{R}^+, V^s),$$

for all $s < 2$.

Notice that this result holds without any restrictions on the size of the parameters α, β and ν. Theorem 4.1 is proved by using Friedrich's method and a priori estimates in H^2. The main theorem of this section is the following one.

Theorem 4.2 *Assume that $\nu > 0$, $\alpha > 0$, $\beta > 0$ and $\alpha < \sqrt{\frac{\nu\beta}{8}}$. There exists a bounded invariant set $\mathcal{A}_2$ of V^2 which attracts every bounded set of V^2 in the H^1-topology.*

To prove this theorem, we first show that, under the restriction $\alpha \leq \sqrt{\frac{\nu\beta}{8}}$, the set

$$\mathcal{W}_2 = \{u \in L^\infty_{loc}(\mathbb{R}^+, V^2) \cap C^0(\mathbb{R}^+, V^s) : u \text{ is a weak solution of } (1.3), u(0) \in V^2\}$$

admits a bounded absorbing set for the H^2-topology.

Proposition 4.1 *Assume that $\nu > 0$, $\alpha > 0$, $\beta > 0$ and $\alpha < \sqrt{\frac{\nu\beta}{8}}$ and $f \in H$, the set of the solutions of (1.3) with initial data in V^2 admits a bounded absorbing set in V^2.*

Proof: Let B be a bounded set of V^2 and $u_0 \in B$. There exists a positive constant R such that $B \subset B_\alpha(R)$, where $B_\alpha(R) = \{u \in V^2 : \|\nabla u\|_{L^2} + \alpha \|\Delta u\|_{L^2} \leq R\}$. Let consider the weak solution $u \in L^\infty_{loc}(\mathbb{R}^+, V^2) \cap C^0(\mathbb{R}^+, V^s)$, $s < 2$, of (1.3) with initial data $u_0 \in B$. For $n \in \mathbb{N}$, we introduce the linear operator Π_n, which is an analogue to the operator J_n defined earlier. For $u \in L^2(\mathbb{T}^2)$, it is given by

$$\Pi_n(u) = \sum_{|k| \leq n} \hat{u}_k e^{-ik.x},$$

with $\hat{u}_k$ the Fourier coefficient of u corresponding to $k \in \mathbb{Z}^2$.

For $n \in \mathbb{N}$, we also consider the regularized system of equations

$$\begin{aligned} &\partial_t (u_n - \alpha\Delta u_n) - \nu\Delta\Pi_n(u_n) + \mathbb{P}\Pi_n (\Pi_n(u_n).\nabla\Pi_n(u_n)) \\ &\qquad\qquad -\alpha\mathbb{P}\Pi_n (\text{div } (L_n^t A_n + A_n L_n + \Pi_n(u_n).\nabla A_n)) \\ &\qquad\qquad\qquad -\beta\mathbb{P}\Pi_n \left(\text{div } \left(|A_n|^2 A_n\right)\right) = \mathbb{P}\Pi_n (f), \qquad (4.1) \\ &\text{div } u_n = 0, \\ &u_{n|t=0} = \Pi_n(u_0) \in V^2, \end{aligned}$$

where $L_n = \nabla\Pi_n(u_n)$, $A = L_n + L_n^t$ and $\mathbb{P}$ is the classical Leray projector.

Via the classical Cauchy-Lipschitz Theorem, we can show that the system (4.1) is well-posed in $\Pi_n V^2$ and admits a unique solution $u_n \in C^1([0, t_n), \Pi_n V^2)$, where $t_n > 0$. In addition, since $\Pi_n^2 = \Pi_n$, we see that $\Pi_n(u_n)$ is also a solution of (4.1). Thus, due to the fact that the solution of (4.1) is unique, one has $\Pi_n(u_n) = u_n$.

Arguing like in [10], we can show that u_n is global in time and converge up to a subsequence to a solution of the system (1.3) when n goes to infinity. Since the solutions of (1.3) are unique, u_n converge to u. More precisely, one has

$$\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weak* in} \quad L^\infty_{loc}\left(\mathbb{R}^+, V^2\right), \\
&u_n(t) \rightharpoonup u(t), \quad \text{weakly in} \quad V^2, \quad \text{for all} \quad t \geq 0, \\
&u_n \to u, \quad \text{strongly in} \quad C^0\left(\mathbb{R}^+, H^s(\mathbb{T}^2)^2\right), \quad \text{for all} \quad 0 \leq s < 2.
\end{aligned}$$

We perform now the L^2-inner product of the first line of (4.1) with $-\Delta u_n$. Using the fact that $\mathbb{P}$ and Π_n are self-adjoint for the L^2-scalar product and the fact that $\Pi_n u_n = \mathbb{P} u_n = u_n$, we notice that

$$\begin{aligned}
\beta\left(\mathbb{P}\Pi_n \text{div }\left(|A_n|^2 A_n\right), \Delta u_n\right)_{L^2} &= \beta\left(\text{div }\left(|A_n|^2 A_n\right), \Delta u_n\right)_{L^2} \\
&= -\frac{\beta}{2}\int_{\mathbb{T}^2} |A_n|^2 A_n \Delta A_n dx \\
&= \frac{\beta}{2}\int_{\mathbb{T}^2} |A_n|^2 |\nabla A_n|^2 dx + \beta \sum_{j=1}^2 \int_{\mathbb{T}^2} \left(A_n : \partial_j A_n\right) dx.
\end{aligned}$$

Consequently, one has

$$\begin{aligned}
\frac{1}{2}\partial_t\left(\|\nabla u_n\|^2_{L^2} + \alpha\|\Delta u_n\|^2_{L^2}\right) + \nu\|\Delta u_n\|^2_{L^2} + \frac{\beta}{2}\int_{\mathbb{T}^2} |A_n|^2 |\nabla A_n|^2 dx \\
+\beta \sum_{j=1}^2 \int_{\mathbb{T}^2} \left(A_n : \partial_j A_n\right)^2 dx = I + J + K + L
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
&I = \left(u_n.\nabla u_n, \Delta u_n\right)_{L^2}, \\
&J = -\alpha\left(\text{div }\left(L^t_n A_n + A_n L_n\right), \Delta u_n\right)_{L^2}, \\
&K = -\alpha\left(\text{div }\left(u_n.\nabla A_n\right), \Delta u_n\right)_{L^2}, \\
&L = -\left(f, \Delta u_n\right)_{L^2}.
\end{aligned}$$

Using the divergence free property of u_n, a short computation implies

$$I = \left(u_n.\nabla u_n, \Delta u_n\right)_{L^2} = 0.$$

We now compute J. Via some integrations by parts, we have

$$\begin{aligned} J &= \frac{\alpha}{2} \sum_{i,j,k=1}^{2} \int_{\mathbb{T}^2} \left(\partial_i u_{n,k} A_n^{k,j} + A_n^{i,j} \partial_j u_{n,k}\right) \Delta A_n^{i,j} dx \\ &= \frac{\alpha}{2} \sum_{i,j,k=1}^{2} \int_{\mathbb{T}^2} A_n^{i,k} A_n^{k,j} \Delta A_n^{i,j} dx \end{aligned}$$

Using now the divergence free property of u_n, it is quite simple to see that J vanishes. Indeed, we obtain

$$\begin{aligned} J = &\frac{\alpha}{2} \sum_{k=1}^{2} \int_{\mathbb{T}^2} A_n^{1,k} A_n^{k,1} \Delta A_n^{1,1} dx + \frac{\alpha}{2} \sum_{k=1}^{2} \int_{\mathbb{T}^2} A_n^{2,k} A_n^{k,2} \Delta A_n^{2,2} dx \\ &\qquad + \alpha \sum_{k=1}^{2} \int_{\mathbb{T}^2} A_n^{1,k} A_n^{k,2} \Delta A_n^{1,2} dx \\ = &\frac{\alpha}{2} \int_{\mathbb{T}^2} A_n^{1,1} A_n^{1,1} \Delta A_n^{1,1} dx + \frac{\alpha}{2} \int_{\mathbb{T}^2} A_n^{1,2} A_n^{1,2} \Delta A_n^{1,1} dx \\ &+ \frac{\alpha}{2} \int_{\mathbb{T}^2} A_n^{2,2} A_n^{2,2} \Delta A_n^{2,2} dx + \frac{\alpha}{2} \int_{\mathbb{T}^2} A_n^{1,2} A_n^{1,2} \Delta A_n^{2,2} dx \\ &+ \alpha \int_{\mathbb{T}^2} \left(A_n^{1,1} + A_n^{2,2}\right) A_n^{1,2} \Delta A_n^{1,2} dx. \end{aligned}$$

Since $A_n^{1,1} = -A_n^{2,2}$, we obtain $J = 0$. It remains to estimate K and L. Integrating by parts and using the divergence free property of u_n, we get

$$\begin{aligned} K &= \frac{\alpha}{2} \int_{\mathbb{T}^2} u_n.\nabla A_n : \Delta A_n dx \\ &= -\frac{\alpha}{2} \sum_{j,k=1}^{2} \int_{\mathbb{T}^2} \partial_k u_{n,j} \partial_j A_n : \partial_k A_n dx \\ &= -\frac{\alpha}{4} \sum_{j,k=1}^{2} \int_{\mathbb{T}^2} A_n^{j,k} \partial_j A_n : \partial_k A_n dx. \end{aligned}$$

Via a Fourier decomposition, we can see that

$$\left\| \sum_{i,j,k=1}^{2} \partial_k A_n^{i,j} \right\|_{L^2} \leq 16 \left\| \Delta u_n \right\|_{L^2}.$$

Using now Hölder and Young inequalities, we obtain

$$\begin{aligned} K &\leq 4\alpha \|\Delta u_n\|_{L^2} \|A_n \nabla A_n\|_{L^2} \\ &\leq \mu\nu \|\Delta u_n\|_{L^2}^2 + \frac{2\alpha^2}{\mu\nu} \int_{\mathbb{T}^2} |A_n|^2 |\nabla A_n|^2 \, dx, \end{aligned}$$

where μ is a positive constant such that $0 < \mu < \frac{1}{2}$ which will be made more precise later.

Likewise, we get

$$L \leq \mu\nu \|\Delta u_n\|_{L^2}^2 + \frac{1}{4\mu\nu} \|f\|_{L^2}^2 .$$

Going back to (4.2), we finally have

$$\begin{aligned} &\frac{1}{2}\partial_t \left(\|\nabla u_n\|_{L^2}^2 + \alpha \|\Delta u_n\|_{L^2}^2\right) + (1-2\mu)\,\nu \|\Delta u_n\|_{L^2}^2 \\ &\qquad + \left(\frac{\beta}{2} - \frac{2\alpha^2}{\mu\nu}\right) \int_{\mathbb{T}^2} |A_n|^2 |\nabla A_n|^2 \, dx \leq \frac{1}{4\mu\nu} \|f\|_{L^2}^2 . \end{aligned} \tag{4.3}$$

If $\alpha < \sqrt{\dfrac{\nu\beta}{8}}$, then there exists $\mu_0 \in \left(0, \frac{1}{2}\right)$ such that $\dfrac{\beta}{2} - \dfrac{2\alpha^2}{\mu_0 \nu} = 0$. Taking, $\mu = \mu_0$, it comes

$$\frac{1}{2}\partial_t \left(\|\nabla u_n\|_{L^2}^2 + \alpha \|\Delta u_n\|_{L^2}^2\right) + C_1 \nu \|\Delta u_n\|_{L^2}^2 \leq \frac{1}{C_2 \nu} \|f\|_{L^2}^2 , \tag{4.4}$$

where C_1 and C_2 are two positive constants.

Since $u_n \in V^2$, the Poincaré inequality implies

$$\|\nabla u_n\|_{L^2} \leq \|\Delta u_n\|_{L^2} .$$

Setting $M = \dfrac{C_1 \nu t}{2\max(\alpha, 1)}$, the inequality (4.4) becomes

$$\frac{1}{2}\partial_t \left(\left(\|\nabla u_n\|_{L^2}^2 + \alpha \|\Delta u_n\|_{L^2}^2\right) e^{Mt}\right) \leq \frac{1}{C_2 \nu} \|f\|_{L^2}^2 e^{Mt}. \tag{4.5}$$

We integrate (4.5) in time between 0 and $t \geq 0$, and we obtain

$$\begin{aligned} \frac{1}{2}\left(\|\nabla u_n(t)\|_{L^2}^2 + \alpha \|\Delta u_n(t)\|_{L^2}^2\right) \leq \frac{1}{2}\left(\|\nabla \Pi_n(u_0)\|_{L^2}^2 + \alpha \|\Delta \Pi_n(u_0)\|_{L^2}^2\right) e^{-Mt} \\ + N \|f\|_{L^2}^2 , \end{aligned} \tag{4.6}$$

where N is a positive constant depending on α and ν, independent of u_0. We pass now to the lim inf when n tends to infinity in the inequality (4.6). Since $u_n(t)$ converge weakly to $u(t)$ in V^2, we obtain

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 + \alpha \|\Delta u(t)\|_{L^2}^2 &\leq \liminf_{n\to\infty} \left(\|\nabla u_n(t)\|_{L^2}^2 + \alpha_1 \|\Delta u_n(t)\|_{L^2}^2\right) \\ &\leq \left(\|\nabla u_0\|_{L^2}^2 + \alpha_1 \|\Delta u_0\|_{L^2}^2\right) e^{-Mt} + 2N \|f\|_{L^2}^2 .\end{aligned}$$

Since $u_0 \in B_\alpha(R)$, we obtain

$$\|\nabla u(t)\|_{L^2}^2 + \alpha \|\Delta u(t)\|_{L^2}^2 \leq R^2 e^{-Mt} + 2N \|f\|_{L^2}^2 . \tag{4.7}$$

We finally take $t_0 \geq -\dfrac{1}{M} \log \left(\dfrac{N \|f\|_{L^2}^2}{R^2} \right)$ and obtain, for all $t \geq t_0$,

$$\|\nabla u(t)\|_{L^2}^2 + \alpha \|\Delta u(t)\|_{L^2}^2 \leq 3N \|f\|_{L^2}^2 . \tag{4.8}$$

Due the Poincaré inequality (2.1), it concludes the proof of this proposition.

□

Proof of Theorem 4.2

Let $\mathcal{W}$ be the generalized semiflow of the weak solutions of (1.3) in V, $\{T(t)\}_{t\geq 0}$ be the family of operators associated to $\mathcal{W}$ given by (3.1) and $\mathcal{W}_2$ be the set of solutions with initial data in V^2. Let B_2 be the bounded absorbing set of $\mathcal{W}_2$ in V^2. We define the ω-limit set $\mathcal{A}_2$ of B_2 for the topology of V, that is to say

$$\begin{aligned}\mathcal{A}_2 = \omega(B_2) = \{z \in V :\ &\text{there exist } \varphi_j \in \mathcal{W} \text{ with } \varphi_j(0) \in B_2, \\ &\left.\text{and a sequence } t_j \in \mathbb{R}^+, t_j \to +\infty, \text{ such that } \varphi_j(t_j) \underset{j\to\infty}{\longrightarrow} z \text{ in } V \right\}.\end{aligned}$$

We start by proving the following lemma, which describes $\mathcal{A}_2$. We also assume that $\nu > 0$, $\alpha > 0$, $\beta > 0$ and $\alpha < \sqrt{\dfrac{\nu\beta}{8}}$.

Lemma 4.1 *The set $\mathcal{A}_2$ is non-empty, contained in B_2 and invariant by T.*

Proof: Since B_2 is a bounded absorbing set for weak solutions with data in V^2, every ball that contains B_2 is also a bounded absorbing set for these solutions. Thus, we can assume that B_2 is a ball of radius $R > 0$, $B_2 = \{u \in V^2 : \|u\|_{H^2} \leq R\}$. Let $u_j \in \mathcal{W}_2$ such that $u_j(0) \in B_2$ and t_j such that $t_j \to +\infty$. Since B_2 is bounded in V^2 and absorbs all the bounded sets of V^2, it is clear that there exists j_0, such that, for all $j \geq j_0$, $u_j(t_j)$ is bounded in V^2 uniformly with respect to j. Due to the compactness of V^2 in V, there exists $z \in V$ and a subsequence of $u_j(t_j)$ that converge to z in V. Consequently, $\mathcal{A}_2$ is non-empty.

The fact $\mathcal{A}_2$ is contained in B_2 is a consequence of the definition of B_2. Indeed, let $z \in \mathcal{A}_2$. There exist $u_j \in \mathcal{W}_2$ with $u_j(0) \in B_2$ and t_j, $t_j \to +\infty$ such that $u_j(t_j) \to z$ in the H^1-topology. Furthermore, $u_j(t_j)$ is bounded in V^2 uniformly with respect to j, and consequently there exists $v \in V^2$ such that, up to a subsequence,

$$u_j(t_j) \rightharpoonup v \text{ weakly in } V^2.$$

Necessarily, $v = z$ and thus $z \in V^2$. Furthermore, the weak convergence implies

$$\|z\|_{H^2} \leq \liminf_{j \to +\infty} \|u_j(t_j)\|_{H^2} \leq R. \tag{4.9}$$

Consequently, $\mathcal{A}_2 \subset B_2$.

It remains to show that $\mathcal{A}_2$ is invariant by T. To show this property, we first show that $\mathcal{A}_2$ is quasi-invariant and then use this property to show that it is actually invariant. Let $z \in \mathcal{A}_2 = \omega(B_2) \subset B_2$. There exists a sequence $u_j \in \mathcal{W}_2$ and $t_j \to +\infty$ such that $u_j(t_j) \to z$ in V. Due to the property $(H2)$, $u_j^{t_j} \in \mathcal{W}_2$ with initial data $u_j(t_j)$. Letting j go to infinity and applying the property $(H4)$, there exists a subsequence of $u_j^{t_j}$ (that we still note $u_j^{t_j}$) and $v_0 \in \mathcal{W}$ with $v_0(0) = z$ such that

$$u_j^{t_j}(t) \to v_0(t), \qquad \text{for all } t \in \mathbb{R}^+. \tag{4.10}$$

Furthermore, the fact that $z \in B_2$ implies that $v_0 \in \mathcal{W}_2$. Besides, from the definition of B_2, it is clear that, if j is sufficiently large, $u_j^{t_j}(t) = u_j(t_j + t) \in B_2$. Thus, the identity (4.10) implies that $v_0(t) \in \mathcal{A}_2 = \omega(B_2)$, for all $t \geq 0$.

We consider now $u_j^{t_j-1}$. We notice that $u_j^{t_j-1}(0) = u_j(t_j - 1)$ which belongs to B_2 if j is sufficiently large and is consequently bounded in V^2, uniformly with respect to j. By the compactness of V^2 in V, one can extract a convergent subsequence of $u_j(t_j - 1)$. By the property $(H4)$, up to a further subsequence, there exists $v_1 \in \mathcal{W}$ such that

$$u_j^{t_j-1}(t) \to v_1(t), \qquad \text{for all } t \in \mathbb{R}^+. \tag{4.11}$$

Since $u_j(t_j - 1)$ is bounded in V^2 and converge to $v_1(0)$ in V, it is clear that $v_1(0) \in V^2$ and consequently $v_1 \in \mathcal{W}_2$. Furthermore, since $u_j^{t_j-1}(t) \in B_2$, then $v_1(t) \in \mathcal{A}_2$, for all

$t \geq 0$. Due to the identities (4.10) and (4.11), we deduce that $v_1^1 = v_0$.

By the same process, we construct for $n \in \mathbb{N}$, $v_n \in \mathcal{W}_2$ such that $v_n(t) \in \mathcal{A}_2$ for all $t \in \mathbb{R}^+$ and $v_n^1 = v_{n-1}$. Then, we define the complete orbit $v \in W_2$ by

$$v(t) = \begin{cases} v_n(t+n), & t \in [-n-1, -n], \\ v_0(t), & t \geq 0. \end{cases}$$

By the concatenation property $(H3)$, v is a complete orbit such that $v(0) = z$. Besides, the properties of v_n imply that $v(t) \in \mathcal{A}_2$, for all $t \in \mathbb{R}$. Thus, $\mathcal{A}_2$ is quasi-invariant.

We show now that $\mathcal{A}_2$ is actually invariant. Let $t_0 \in \mathbb{R}^+$ and $v \in T(t_0)\mathcal{A}_2$. According to the definition of T, there exists $z \in \mathcal{A}_2$ and $w \in \mathcal{W}_2$ such that $w(0) = z$ and $w(t_0) = v$. Let $\varphi \in \mathcal{W}_2$ be a complete orbit such that $\varphi(0) = z$. By the concatenation property, we define

$$\psi(t) = \begin{cases} \varphi(t), & t \leq 0, \\ w(t), & t \geq 0. \end{cases}$$

Then, we set $\psi_n(t) = \psi(t-n)$. In particular $\psi_n \in \mathcal{W}$ and $\psi_n(t_0+n) = \psi(t_0) = v$. Furthermore, due to the quasi-invariance of $\mathcal{A}_2$, $\psi_n(0) = \psi(-n) \in \mathcal{A}_2 \subset B_2$. Consequently, $v \in \omega(B_2)$ and thus $T(t_0)\mathcal{A}_2 \subset \mathcal{A}_2$, that is to say that $\mathcal{A}_2$ is positively invariant.

Reversely, the quasi-invariance of $\mathcal{A}_2$ implies that $\mathcal{A}_2 \subset T(t)\mathcal{A}_2$, for all $t \in R^+$. In fact, let $z \in \mathcal{A}_2$ and φ be a complete orbit in $\mathcal{A}_2$ such that $\varphi(0) = z$. Defining $\varphi_{t_0}(t) = \varphi(t-t_0)$, we see that $\varphi_{t_0}(t_0) = z$. Thus, one has $z \in T(t_0)\varphi_{t_0}(0)$. Since $\varphi_{t_0}(0) \in \mathcal{A}_2$, then $z \in T(t_0)\mathcal{A}_2$ and $\mathcal{A}_2$ is consequently invariant.

□

To finish the proof of Theorem 4.2, we show by a contradiction argument that the set $\mathcal{A}_2$ attracts B_2 for the H^1-topology. Since every bounded set of V^2 is absorbed by B_2, it would imply that every bounded set of V^2 is attracted by $\mathcal{A}_2$.

Lemma 4.2 *The set $\mathcal{A}_2$ attracts all the bounded sets of V^2 in the H^1-topology.*

Proof: The proof of this lemma is nearly obvious, and is obtained through a contradiction argument. Let B_2 be the bounded absorbing set of $\mathcal{W}_2$. Assume by contradiction that $\mathcal{A}_2$ does not attract B_2. Then, there exist $\varepsilon > 0$, $u_j \in \mathcal{W}_2$ with initial data in B_2 and t_j, $t_j \to +\infty$, such that

$$\inf_{z \in \mathcal{A}_2} \|u_j(t_j) - z\|_{H^1} \geq \varepsilon, \quad \text{for all} \quad j \in \mathbb{N}. \tag{4.12}$$

Since B_2 is a bounded absorbing set in the H^2-topology, there exists $j_0 \in \mathbb{N}$ such that $u_j(t_j) \in B_2$, for all $j \geq j_0$. Consequently, for all $j \geq j_0$, $u_j(t_j)$ is bounded in V^2, uniformly with respect to j. Due to the compactness of V^2 in V, there exists $z \in V$ and a subsequence of $u_j(t_j)$ (that we still note $u_j(t_j)$) such that $u_j(t_j) \to z$ for the H^1--topology. Furthermore, the fact that $u_j(0) \in B_2$ implies that $z \in \mathcal{A}_2$, which contradicts (4.12). Consequently $\mathcal{A}_2$ attracts B_2 in the H^1-topology.

Let B be a bounded set of V^2. There exists $t_0 \geq 0$ such that $T(t)B \subset B_2$, for all $t \geq t_0$. Since $\mathcal{A}_2$ attracts B_2, it attracts also $T(t)B$ for all $t \geq t_0$, and consequently B.

$\square$

The lemmas 4.1 and 4.2 imply Theorem 4.2.

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